International School of Economics at TSU
Econometrics 2
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Problem Set 1 - Review of Probability and Statistics

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1 Consider a sample space Ω comprising four possible outcomes: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

Consider also the three events E, F, and G defined as follows:

$$E = \{\omega_1\}, \quad F = \{\omega_1, \omega_2\}, \quad G = \{\omega_1, \omega_2, \omega_3\}$$

Suppose the probabilities of the events are given as:

$$P(E) = \frac{1}{10}, \quad P(F) = \frac{5}{10}, \quad P(G) = \frac{7}{10}$$

Now, consider a fourth event H defined as: $H = \{\omega_2, \omega_4\}$. Find $P(H)$.

Solution

First note that, by additivity,

$$P(H) = P(\{\omega_2\} \cup \{\omega_4\}) = P(\{\omega_2\}) + P(\{\omega_4\})$$

Therefore, in order to compute $P(H)$, we need to compute $P(\{\omega_2\})$ and $P(\{\omega_4\})$.
$P(\{\omega_2\})$ is found using additivity on F:

$$\frac{5}{10} = P(F) = P(\{\omega_1\} \cup \{\omega_2\}) = P(\{\omega_1\}) + P(\{\omega_2\})$$

$$= P(E) + P(\{\omega_2\}) = \frac{1}{10} + P(\{\omega_2\})$$

so that

$$P(\{\omega_2\}) = \frac{5}{10} - \frac{1}{10} = \frac{4}{10}$$

$P(\{\omega_4\})$ is found using the fact that one minus the probability of an event is equal to the probability of its complement and the fact that $\{\omega_4\} = G^c$:

$$P(\{\omega_4\}) = P(G^c) = 1 - P(G) = 1 - \frac{7}{10} = \frac{3}{10}$$

As a consequence,

$$P(H) = P(\{\omega_2\}) + P(\{\omega_4\}) = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$$

Problem 2 Let E and F be two events. Let E^c be a zero-probability event and $P(F) = \frac{1}{2}$. Compute $P(E \cap F)$.

Solution

E^c is a zero-probability event, which means that

$$P(E^c) = 0$$

Furthermore, using the formula for the probability of a complement, we obtain

$$P(E) = 1 - P(E^c) = 1 - 0 = 1$$

It is also true that

$$P(E \cap F) = P(E) + P(F) - P(E \cup F)$$

$$= 1 + \frac{1}{2} - P(E \cup F) = \frac{3}{2} - P(E \cup F)$$

Since $(E \cup F) \supseteq E$, by monotonicity, we obtain

$$P(E \cup F) \geq P(E)$$

Since $P(E) = 1$ and probabilities cannot be greater than 1, this implies

$$P(E \cup F) = 1$$

Thus, putting pieces together, we get

$$P(E \cap F) = \frac{3}{2} - P(E \cup F) = \frac{3}{2} - 1 = \frac{1}{2}$$

Problem 3 A firm undertakes two projects, A and B. The probabilities of having a successful outcome are $\frac{2}{3}$ for project A and $\frac{4}{5}$ for project B. What is the probability that neither of the two projects will have a successful outcome if their outcomes are independent?

Solution

Denote by E the event "project A is successful", by F the event "project B is successful" and by G the event "neither of the two projects is successful". The event G can be expressed as

$$G = E^c \cap F^c$$

where E^c and F^c are the complements of E and F. Thus, the probability that neither of the two projects will have a successful outcome is

$$P(G) = P(E^c \cap F^c)$$

$$\text{(A)} \quad = P((E \cup F)^c)$$

$$\text{(B)} \quad = 1 - P(E \cup F)$$

$$\text{(C)} \quad = 1 - (P(E) + P(F) - P(E \cap F))$$

$$= 1 - P(E) - P(F) + P(E \cap F)$$

$$\text{(D)} \quad = 1 - P(E) - P(F) + P(E)P(F)$$

$$= 1 - \frac{2}{3} - \frac{4}{5} + \frac{2}{3} \cdot \frac{4}{5}$$

$$= 1 - \frac{2}{3} - \frac{4}{5} + \frac{8}{15}$$

$$= \frac{15 - 10 - 12 + 8}{15} = \frac{1}{15}$$

where: in step A we have used De Morgan's law; in step B we have used the formula for the probability of a complement; in step C we have used the formula for the probability of a union; in step D we have used the fact that E and F are independent.

Problem 4 An economics consulting firm has created a model to predict recessions. The model predicts a recession with probability 80% when a recession is indeed coming and with probability 10% when no recession is coming. The unconditional probability of falling into a recession is 20%. If the model predicts a recession, what is the probability that a recession will indeed come?

Solution

What we know about this problem can be formalized as follows:

$$P(\text{rec. pred.} \mid \text{rec. coming}) = \frac{8}{10}$$

$$P(\text{rec. pred.} \mid \text{rec. not coming}) = \frac{1}{10}$$

$$P(\text{rec. coming}) = \frac{2}{10}$$

$$P(\text{rec. not coming}) = 1 - P(\text{rec. coming}) = 1 - \frac{2}{10} = \frac{8}{10}$$

The unconditional probability of predicting a recession can be derived using the law of total probability:

$$P(\text{rec. pred.}) = P(\text{rec. pred.} \mid \text{rec. coming})P(\text{rec. coming}) + P(\text{rec. pred.} \mid \text{rec. not coming})P(\text{rec. not coming})$$

$$= \frac{8}{10} \cdot \frac{2}{10} + \frac{1}{10} \cdot \frac{8}{10} = \frac{24}{100}$$

Using Bayes' rule we obtain:

$$P(\text{rec. coming} \mid \text{rec. pred.}) = \frac{P(\text{rec. pred.} \mid \text{rec. coming})P(\text{rec. coming})}{P(\text{rec. pred.})}$$

$$= \frac{\frac{8}{10} \cdot \frac{2}{10}}{\frac{24}{100}} = \frac{16}{100} \cdot \frac{100}{24} = \frac{2}{3}$$

Problem 5 Let X be a discrete random variable. Let its support R_X be

$$R_X = \{0, 1, 2, 3\}.$$

Let its probability mass function $p_X(x)$ be

$$p_X(x) = \begin{cases} \binom{3}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x} & \text{if } x \in R_X, \\ 0 & \text{if } x \notin R_X, \end{cases}$$

where

$$\binom{3}{x} = \frac{3!}{x!(3-x)!}$$

is a binomial coefficient.

Calculate the probability

$$P(X < 3).$$

Solution

First note that, by additivity, we have

$$P(X < 3) = P(\{X = 0\} \cup \{X = 1\} \cup \{X = 2\})$$

$$= P(\{X = 0\}) + P(\{X = 1\}) + P(\{X = 2\})$$

$$= p_X(0) + p_X(1) + p_X(2)$$

Therefore, in order to compute $P(X < 3)$, we need to evaluate the probability mass function at the three points $x = 0$, $x = 1$ and $x = 2$:

$$p_X(0) = \binom{3}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{3-0} = \frac{3!}{0!3!}\cdot 1\cdot\frac{27}{64} = \frac{27}{64}$$

$$p_X(1) = \binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{3-1} = \frac{3!}{1!2!}\cdot\frac{1}{4}\cdot\frac{9}{16}$$

$$= 3\cdot\frac{1}{4}\cdot\frac{9}{16} = \frac{27}{64}$$

$$p_X(2) = \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{3-2} = \frac{3!}{2!1!}\cdot\frac{1}{16}\cdot\frac{3}{4} = \frac{9}{64}$$

Finally,

$$P(X < 3) = p_X(0) + p_X(1) + p_X(2)$$

$$= \frac{27}{64} + \frac{27}{64} + \frac{9}{64} = \frac{63}{64}$$

Problem 6 Let X be an absolutely continuous random variable. Let its support R_X be

$$R_X = [0, 1].$$

Let its probability density function $f_X(x)$ be

$$f_X(x) = \begin{cases} 1 & \text{if } x \in R_X, \\ 0 & \text{if } x \notin R_X. \end{cases}$$

Compute

$$P\left(\frac{1}{2} \leq X \leq 2\right).$$

Solution

The probability that an absolutely continuous random variable takes a value in a given interval is equal to the integral of the probability density function over that interval:

$$P\left(\frac{1}{2} \leq X \leq 2\right) = P\left(X \in \left[\frac{1}{2}, 2\right]\right) = \int_{1/2}^{2} f_X(x)\,dx$$

$$= \int_{1/2}^{1} dx = [x]_{1/2}^{1} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Problem 7 Let X be an absolutely continuous random variable with uniform distribution on the interval $[1, 3]$.

Its support is

$$R_X = [1, 3]$$

Its probability density function is

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [1,3], \\ 0 & \text{otherwise} \end{cases}$$

Compute the expected value of X.

Solution

Since X is absolutely continuous, its expected value can be computed as an integral:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{-\infty}^{1} x f_X(x)\,dx + \int_{1}^{3} x f_X(x)\,dx + \int_{3}^{\infty} x f_X(x)\,dx$$

$$= \int_{-\infty}^{1} x \cdot 0\,dx + \int_{1}^{3} x \cdot \frac{1}{2}\,dx + \int_{3}^{\infty} x \cdot 0\,dx$$

$$= 0 + \frac{1}{2}\int_1^3 x\,dx + 0$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^3$$

$$= \frac{1}{2}\left(\frac{1}{2}\cdot 3^2 - \frac{1}{2}\cdot 1^2\right) = \frac{1}{2}\left(\frac{9}{2} - \frac{1}{2}\right)$$

$$= \frac{1}{2}\cdot 4 = 2$$

Problem 8 Let X and Y be two random variables, having expected values

$$\mathbb{E}[X] = \sqrt{2}$$

$$\mathbb{E}[Y] = 1$$

Compute the expected value of the random variable Z defined as follows:

$$Z = \sqrt{2}X + Y$$

Solution

Using the linearity of the expected value operator, we obtain

$$\mathbb{E}[Z] = \mathbb{E}[\sqrt{2}X + Y] = \sqrt{2}\,\mathbb{E}[X] + \mathbb{E}[Y]$$

$$= \sqrt{2}\,\sqrt{2} + 1 = 2 + 1 = 3.$$

Problem 9 Let X be a discrete random variable with support

$$R_X = \{0, 1, 2, 3\}$$

and probability mass function

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x \in R_X, \\ 0 & \text{otherwise.} \end{cases}$$

Compute its variance.

Solution

The expected value of X is

$$\mathbb{E}[X] = \sum_{x \in R_X} x p_X(x)$$

$$= 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2) + 3 \cdot p_X(3)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$$

The expected value of X^2 is

$$\mathbb{E}[X^2] = \sum_{x \in R_X} x^2 p_X(x)$$

$$= 0^2 \cdot p_X(0) + 1^2 \cdot p_X(1) + 2^2 \cdot p_X(2) + 3^2 \cdot p_X(3)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} + 9 \cdot \frac{1}{4} = \frac{14}{4} = \frac{7}{2}$$

The variance of X is

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{7}{2} - \left(\frac{3}{2}\right)^2$$

$$= \frac{14}{4} - \frac{9}{4} = \frac{5}{4}.$$

Problem 10 Let X be an absolutely continuous random variable with support

$$R_X = [0,1]$$

and probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Compute its variance.

Solution

The expected value of X is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x \cdot 3x^2\, dx = \int_0^1 3x^3\, dx$$

$$= \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4} - 0 = \frac{3}{4}.$$

The expected value of X^2 is

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2 \cdot 3x^2\, dx = \int_0^1 3x^4\, dx$$

$$= \left[\frac{3}{5}x^5\right]_0^1 = \frac{3}{5}.$$

The variance of X is

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2$$

$$= \frac{3}{5} - \frac{9}{16} = \frac{48-45}{80} = \frac{3}{80}.$$

Problem 11 Let X and Y be two random variables such that

$$\mathrm{Var}[X] = 2$$

$$\mathrm{Cov}[X, Y] = 1$$

Compute the covariance

$$\mathrm{Cov}[5X, 2X + 3Y].$$

Solution

By exploiting the bilinearity of the covariance operator, we obtain

$$\mathrm{Cov}[5X, 2X + 3Y] = 5\,\mathrm{Cov}[X, 2X + 3Y]$$

$$= 10\,\mathrm{Cov}[X, X] + 15\,\mathrm{Cov}[X, Y]$$

$$= 10\,\mathrm{Var}[X] + 15\,\mathrm{Cov}[X, Y]$$

$$= 10 \cdot 2 + 15 \cdot 1 = 35.$$

Problem 12 Let $[X \quad Y]$ be a random vector with support

$$R_{XY} = \{[2 \quad 2], [2 \quad 0], [1 \quad 2], [0 \quad 2]\}$$

and joint probability mass function

$$p_{XY}(x, y) = \begin{cases} \frac{1}{4} & \text{if } x = 2 \text{ and } y = 2 \\ \frac{1}{4} & \text{if } x = 2 \text{ and } y = 0 \\ \frac{1}{4} & \text{if } x = 1 \text{ and } y = 2 \\ \frac{1}{4} & \text{if } x = 0 \text{ and } y = 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the conditional expectation of X given $Y = 2$?

Solution

Let us compute the conditional probability mass function of X given $Y = 2$. The marginal probability mass function of Y evaluated at $y = 2$ is

$$p_Y(2) = \sum_{\{(x,y)\in R_{XY}: y=2\}} p_{XY}(x, y) = p_{XY}(2, 2) + p_{XY}(1, 2) + p_{XY}(0, 2) = \frac{3}{4}$$

The support of X is

$$R_X = \{0, 1, 2\}$$

Thus, the conditional probability mass function of X given $Y = 2$ is

$$p_{X|Y=2}(x) = \begin{cases} \frac{p_{XY}(0,2)}{p_Y(2)} = \frac{1/4}{3/4} = \frac{1}{3} & \text{if } x = 0 \\ \frac{p_{XY}(1,2)}{p_Y(2)} = \frac{1/4}{3/4} = \frac{1}{3} & \text{if } x = 1 \\ \frac{p_{XY}(2,2)}{p_Y(2)} = \frac{1/4}{3/4} = \frac{1}{3} & \text{if } x = 2 \\ 0 & \text{if } x \notin R_X \end{cases}$$

The conditional expectation of X given $Y = 2$ is

$$\mathbb{E}[X \mid Y = 2] = 0 \cdot p_{X|Y=2}(0) + 1 \cdot p_{X|Y=2}(1) + 2 \cdot p_{X|Y=2}(2)$$

$$= 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 1$$