International School of Economics at TSU

Econometrics 2

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Problem Set 2

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1: Let $X_1, \ldots, X_n$ be n mutually independent standard normal random variables. Let $b \in (0,1)$ be a constant. Find the distribution of the random variable Y defined as

$$Y = \sum_{i=1}^{n} b^i X_i$$

Problem 2 Let $\{X_n\}$ be an IID sequence of continuous random variables having a uniform distribution with support

$$R_{X_n} = \left[-\frac{1}{n}, \frac{1}{n}\right]$$

and probability density function

$$f_{X_n}(x) = \begin{cases} \frac{n}{2} & \text{if } x \in [-\frac{1}{n}, \frac{1}{n}] \\ 0 & \text{if } x \notin [-\frac{1}{n}, \frac{1}{n}] \end{cases}$$

Find the probability limit (if it exists) of the sequence $\{X_n\}$.

Problem 3 Let U be a random variable with a uniform distribution on $[0,1]$. That is, U is continuous with support

$$R_U = [0,1]$$

and probability density function:

$$f_U(u) = \begin{cases} 1 & \text{if } u \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Define a sequence of random variables $\{X_n\}$ as follows:

$$\begin{aligned}
&X_1 = \mathbb{1}_{\{U\in[0,1]\}}, \quad X_2 = \mathbb{1}_{\{U\in[0,1/2]\}}, \quad X_3 = \mathbb{1}_{\{U\in[1/2,1]\}}, \\
&X_4 = \mathbb{1}_{\{U\in[0,1/4]\}}, \quad X_5 = \mathbb{1}_{\{U\in[1/4,2/4]\}}, \quad X_6 = \mathbb{1}_{\{U\in[2/4,3/4]\}}, \quad X_7 = \mathbb{1}_{\{U\in[3/4,1]\}}, \\
&X_8 = \mathbb{1}_{\{U\in[0,1/8]\}}, \quad X_9 = \mathbb{1}_{\{U\in[1/8,2/8]\}}, \quad X_{10} = \mathbb{1}_{\{U\in[2/8,3/8]\}}, \quad \dots \\
&X_{16} = \mathbb{1}_{\{U\in[0,1/16]\}}, \quad X_{17} = \mathbb{1}_{\{U\in[1/16,2/16]\}}, \quad X_{18} = \mathbb{1}_{\{U\in[2/16,3/16]\}}, \quad \dots
\end{aligned}$$

where $\mathbb{1}_{\{U\in[a,b]\}}$ is the indicator function of the event $\{U \in [a,b]\}$.

Find the probability limit (if it exists) of the sequence $\{X_n\}$.

Problem 4 Let $\{X_n\}$ be a sequence of random variables having distribution functions

$$F_n(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{n}{2n+1}x + \frac{1}{4n+2}x^2 & \text{if } 0 < x \le 1 \\ \frac{n}{2n+1}x - \frac{1}{4n+2}(x^2 - 4x + 2) & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

Find the limit in distribution (if it exists) of the sequence $\{X_n\}$.

Problem 5 Let $\{X_n\}$ be a sequence of random variables having distribution function:

$$F_n(x) = \begin{cases} 0 & \text{if } x < 0 \\ nx & \text{if } 0 \le x \le 1/n \\ 1 & \text{if } x > 1/n \end{cases}$$

Find the limit in distribution (if it exists) of the sequence $\{X_n\}$.

Problem 6 Let $X_1, \dots, X_n$ be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Show that

a. $\mathbb{E}[\bar{X}] = \mu$,

b. $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$,

c. $\mathbb{E}[S^2] = \sigma^2$ where $S^2 = \frac{1}{n-1}\left[\sum_{i=1}^n X_i^2 - n\bar{X}^2\right]$

Problem 7 Let $X_1, \dots, X_n$ be a random sample from a population with mean μ and variance σ^2. Show that

$$\mathbb{E}[\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}] = 0 \quad \text{and} \quad \text{Var}(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}) = 1.$$

Problem 8 Let $\{y_i : i = 1, 2, \dots\}$ be an independent, identically distributed sequence with $\mathbb{E}(y_i^2) < \infty$. Let $\mu = \mathbb{E}(y_i)$ and $\sigma^2 = \text{Var}(y_i)$.

a. Let $\bar{y}_N$ denote the sample average based on a sample size of N. Find $\text{Var}[\sqrt{N}(\bar{y}_N - \mu)]$.

b. What is the asymptotic variance of $\sqrt{N}(\bar{y}_N - \mu)$?

c. What is the asymptotic variance of $\bar{y}_N$? Compare this with $\text{Var}(\bar{y}_N)$.

d. What is the asymptotic standard deviation of $\bar{y}_N$?

e. How would you obtain the asymptotic standard error of $\bar{y}_N$?

Problem 9 Let Y be a binomial random variable with parameters $n = 100$ and $p = \frac{1}{2}$.

Using the Central Limit Theorem, show that a normal random variable X with mean $\mu = 50$ and variance $\sigma^2 = 25$ can be used to approximate Y.

Problem 10 Let X be an integrable[1] random variable defined on a sample space Ω. Let X be a positive random variable[2]. Let $c \in \mathbb{R}_{++}$. Prove the following inequality, called **Markov's inequality**:

$$\mathbb{P}(X \geq c) \leq \frac{\mathbb{E}[X]}{c}$$

[1]That is, $\mathbb{E}[|X|] < \infty$.
[2]That is, $X(\omega) \geq 0$ for all $\omega \in \Omega$.

Problem 11 Let X be a square integrable[3] random variable defined on a sample space Ω. Let μ and σ^2 denote the mean and variance of X respectively. Let $k \in \mathbb{R}_{++}$. Prove the following inequality, called **Chebyshev's inequality**:

$$\mathbb{P}(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Problem 12 Let X be an integrable random variable. Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function such that

$$Y = g(X)$$

is also integrable. Prove the following inequality, called **Jensen's inequality**:

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$$

Getting Started with R: 'swirl' Interactive Tutorials

`swirl` is an **R package that interactively teaches you how to use R** directly inside the **console**.

> **Note**
>
> `swirl` works **inside RStudio only**. However, the **R coding skills you learn are fully transferable** to other environments such as **VS Code with the R kernel**, which we will also use during the course.

Installation and Setup

Open **RStudio** and run the following commands in the **Console**:

```r
# Install swirl package
install.packages("swirl")
```

[3]That is, $\mathbb{E}[(X - \mu)^2] < \infty$.

```r
# Load swirl package
library(swirl)
```

```r
# Install swirl tutorials
install_course("R Programming")
install_course("Getting and Cleaning Data")
install_course("Advanced R Programming")
```

> **Note**
>
> The sequence matters. Start with the easiest tutorial and proceed step-by-step. Each tutorial builds on concepts introduced in the previous one.

```r
# Start swirl
swirl()
```

Happy coding!