International School of Economics at TSU
Econometrics 2
Lasha Chochua

Problem Set 2

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1:

Let $X_1, \dots, X_n$ be n mutually independent standard normal random variables. Let $b \in (0, 1)$ be a constant. Find the distribution of the random variable Y defined as

$$Y = \sum_{i=1}^{n} b^i X_i$$

Solution

Being a linear combination of mutually independent normal random variables, Y has a normal distribution with mean

$$\mathbb{E}[Y] = \sum_{i=1}^{n} b^i \mathbb{E}[X_i] = \sum_{i=1}^{n} b^i \cdot 0 = 0$$

and variance

$$\begin{aligned}\operatorname{Var}[Y] &= \sum_{i=1}^{n} \left(b^i\right)^2 \operatorname{Var}[X_i] \\ &= \sum_{i=1}^{n} \left(b^2\right)^i \cdot 1\end{aligned}$$

$$\boxed{A} \quad = \quad \sum_{i=1}^{n} c^i$$

$$
\begin{aligned}
&= c + c^2 + \cdots + c^n \\
&= c(1 + c + \cdots + c^{n-1}) \\
&= c\left(1 + c + \cdots + c^{n-1}\right)\frac{1-c}{1-c} \\
&= \frac{c}{1-c}\left(1 - c^n\right)
\end{aligned}
$$

$$
= \frac{c - c^{n+1}}{1-c} = \frac{b^2 - b^{2n+2}}{1-b^2}
$$

where: in step $\boxed{A}$ we have defined $c = b^2$.

Problem 2

Let $\{X_n\}$ be an IID sequence of continuous random variables having a uniform distribution with support

$$
R_{X_n} = \left[-\frac{1}{n}, \frac{1}{n}\right]
$$

and probability density function

$$
f_{X_n}(x) = \begin{cases} \frac{n}{2} & \text{if } x \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 0 & \text{if } x \notin \left[-\frac{1}{n}, \frac{1}{n}\right] \end{cases}
$$

Find the probability limit (if it exists) of the sequence $\{X_n\}$.

Solution

As n tends to infinity, the probability density tends to become concentrated around the point $x = 0$. Therefore, it seems reasonable to conjecture that the sequence $\{X_n\}$ converges in probability to the constant random variable

$$
X(\omega) = 0, \quad \forall \omega \in \Omega
$$

To rigorously verify this claim we need to use the formal definition of convergence in probability. For any $\varepsilon > 0$,

$$
\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = \lim_{n\to\infty} P(|X_n - 0| > \varepsilon)
$$

$$
\begin{aligned}
&= \lim_{n\to\infty} [1 - P(-\varepsilon \le X_n \le \varepsilon)] \\
&= 1 - \lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} f_{X_n}(x)\, dx \\
&= 1 - \lim_{n\to\infty} \int_{\max(-\varepsilon,-1/n)}^{\min(\varepsilon,1/n)} \frac{n}{2}\, dx
\end{aligned}
$$

$$
\boxed{A} \quad = \quad 1 - \lim_{n\to\infty} \int_{-1/n}^{1/n} \frac{n}{2}\, dx
$$

$$
= 1 - \lim_{n\to\infty} 1 = 0
$$

where: in step $\boxed{A}$ we have used the fact that $\frac{1}{n} < \varepsilon$ when n becomes large.

Problem 3

Let U be a random variable with a uniform distribution on $[0,1]$. That is, U is continuous with support

$$
R_U = [0,1]
$$

and probability density function:

$$
f_U(u) = \begin{cases} 1 & \text{if } u \in [0,1] \\ 0 & \text{otherwise} \end{cases}
$$

Define a sequence of random variables $\{X_n\}$ as follows:

$$
\begin{aligned}
&X_1 = \mathbb{1}_{\{U\in[0,1]\}}, \quad X_2 = \mathbb{1}_{\{U\in[0,1/2]\}}, \quad X_3 = \mathbb{1}_{\{U\in[1/2,1]\}}, \\
&X_4 = \mathbb{1}_{\{U\in[0,1/4]\}}, \quad X_5 = \mathbb{1}_{\{U\in[1/4,2/4]\}}, \quad X_6 = \mathbb{1}_{\{U\in[2/4,3/4]\}}, \quad X_7 = \mathbb{1}_{\{U\in[3/4,1]\}}, \\
&X_8 = \mathbb{1}_{\{U\in[0,1/8]\}}, \quad X_9 = \mathbb{1}_{\{U\in[1/8,2/8]\}}, \quad X_{10} = \mathbb{1}_{\{U\in[2/8,3/8]\}}, \quad \ldots \\
&X_{16} = \mathbb{1}_{\{U\in[0,1/16]\}}, \quad X_{17} = \mathbb{1}_{\{U\in[1/16,2/16]\}}, \quad X_{18} = \mathbb{1}_{\{U\in[2/16,3/16]\}}, \quad \ldots
\end{aligned}
$$

where $\mathbb{1}_{\{U\in[a,b]\}}$ is the indicator function of the event $\{U \in [a,b]\}$.

Find the probability limit (if it exists) of the sequence $\{X_n\}$.

Solution

A generic term X_n of the sequence, being an indicator function, can only take two values:

- It can take value 1 with probability:

$$\mathbb{P}(X_n = 1) = \mathbb{P}\left(U \in \left[\frac{j}{m}, \frac{j+1}{m}\right]\right) = \frac{1}{m}$$

where m satisfies $\frac{n}{2} < m \leq n$ and j satisfies $n = m + j$

- It can take value 0 with probability:

$$\mathbb{P}(X_n = 0) = 1 - \mathbb{P}(X_n = 1) = 1 - \frac{1}{m}$$

Since $m \to \infty$ as $n \to \infty$, we get:

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0) = \lim_{m\to\infty} \left(1 - \frac{1}{m}\right) = 1$$

Therefore, the probability that $X_n = 0$ converges to 1 as $n \to \infty$.
So $\{X_n\}$ converges in probability to the constant random variable:

$$X(\omega) = 0, \quad \forall \omega \in \Omega$$

For any $\varepsilon > 0$:

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) &= \lim_{n\to\infty} \mathbb{P}(|X_n - 0| > \varepsilon) \\
\boxed{\text{A}} \quad &= \lim_{n\to\infty} \mathbb{P}(X_n > \varepsilon) \\
\boxed{\text{B}} \quad &= \lim_{n\to\infty} \mathbb{P}(X_n = 1) \\
&= \lim_{m\to\infty} \frac{1}{m} = 0
\end{aligned}$$

- Step $\boxed{\text{A}}$: Used $X_n \geq 0$
- Step $\boxed{\text{B}}$: Used that $X_n \in \{0, 1\}$

Hence, $\{X_n\} \xrightarrow{p} 0$

Problem 4

Let $\{X_n\}$ be a sequence of random variables having distribution functions

$$F_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{n}{2n+1}x + \frac{1}{4n+2}x^2 & \text{if } 0 < x \leq 1 \\ \frac{n}{2n+1}x - \frac{1}{4n+2}(x^2 - 4x + 2) & \text{if } 1 < x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

Find the limit in distribution (if it exists) of the sequence $\{X_n\}$.

Solution

If $0 < x \leq 1$, then

$$\begin{aligned} \lim_{n\to\infty} F_n(x) &= \lim_{n\to\infty} \left[\frac{n}{2n+1}x + \frac{1}{4n+2}x^2\right] \\ &= x \cdot \lim_{n\to\infty}\left(\frac{n}{2n+1}\right) + x^2 \cdot \lim_{n\to\infty}\left(\frac{1}{4n+2}\right) \\ &= x \cdot \frac{1}{2} + x^2 \cdot 0 = \frac{1}{2}x \end{aligned}$$

If $1 < x \leq 2$, then

$$\begin{aligned} \lim_{n\to\infty} F_n(x) &= \lim_{n\to\infty} \left[\frac{n}{2n+1}x - \frac{1}{4n+2}(x^2 - 4x + 2)\right] \\ &= x \cdot \lim_{n\to\infty}\left(\frac{n}{2n+1}\right) + (x^2 - 4x + 2) \cdot \lim_{n\to\infty}\left(\frac{-1}{4n+2}\right) \\ &= x \cdot \frac{1}{2} + (x^2 - 4x + 2) \cdot 0 = \frac{1}{2}x \end{aligned}$$

We now need to verify that the function

$$F_X(x) = \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{2}x & \text{if } 0 < x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

is a proper distribution function. The function is increasing, continuous, its limit at minus infinity is 0 and its limit at plus infinity is 1, hence it satisfies the four properties that a proper distribution function must satisfy. This implies that $\{X_n\}$ converges in distribution to a random variable X having distribution function $F_X(x)$.

Problem 5

Let $\{X_n\}$ be a sequence of random variables having distribution function:

$$F_n(x) = \begin{cases} 0 & \text{if } x < 0 \\ nx & \text{if } 0 \leq x \leq 1/n \\ 1 & \text{if } x > 1/n \end{cases}$$

Find the limit in distribution (if it exists) of the sequence $\{X_n\}$.

Solution

The distribution functions $F_n(x)$ converge to the function

$$G_X(x) = \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

This is the cumulative distribution function of a **degenerate** random variable X concentrated at zero.

Hence, the sequence $\{X_n\}$ converges in distribution to a random variable X with distribution function

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Problem 6

Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Show that

a. $\mathbb{E}\bar{X} = \mu$,

b. $\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$,

c. $\mathbb{E}S^2 = \sigma^2$ where $S^2 = \frac{1}{n-1}\left[\sum_{i=1}^n X_i^2 - n\bar{X}^2\right]$

Solution

a.

$$\mathbb{E}\bar{X} = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n}\mathbb{E}\left(\sum_{i=1}^n X_i\right) = \frac{1}{n}n\mathbb{E}X_1 = \mu.$$

b.

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \frac{1}{n^2}n\mathrm{Var}X_1 = \frac{\sigma^2}{n}.$$

c.

$$\mathbb{E}S^2 = \mathbb{E}\left(\frac{1}{n-1}\left[\sum_{i=1}^n X_i^2 - n\bar{X}^2\right]\right)$$

$$= \frac{1}{n-1}\left(n\mathbb{E}X_1^2 - n\mathbb{E}\bar{X}^2\right)$$

$$= \frac{1}{n-1}\left(n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right) = \sigma^2 \quad \text{use b here.}$$

Problem 7

Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and variance σ^2. Show that

$$\mathbb{E}\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = 0 \quad \text{and} \quad \mathrm{Var}\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = 1.$$

Solution

Using $\mathbb{E}\bar{X}_n = \mu$ and $\mathrm{Var}(\bar{X}_n) = \sigma^2/n$, we obtain

$$\mathbb{E}\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sqrt{n}}{\sigma}\mathbb{E}(\bar{X}_n - \mu) = \frac{\sqrt{n}}{\sigma}(\mu - \mu) = 0.$$

$$\mathrm{Var}\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{n}{\sigma^2}\mathrm{Var}(\bar{X}_n - \mu) = \frac{n}{\sigma^2}\mathrm{Var}(\bar{X}_n) = \frac{n}{\sigma^2}\frac{\sigma^2}{n} = 1.$$

Problem 8

Let $\{y_i : i = 1, 2, \dots\}$ be an independent, identically distributed sequence with $\mathbb{E}(y_i^2) < \infty$. Let $\mu = \mathbb{E}(y_i)$ and $\sigma^2 = \mathrm{Var}(y_i)$.

a. Let $\bar{y}_N$ denote the sample average based on a sample size of N. Find $\mathrm{Var}[\sqrt{N}(\bar{y}_N - \mu)]$.

b. What is the asymptotic variance of $\sqrt{N}(\bar{y}_N - \mu)$?

c. What is the asymptotic variance of $\bar{y}_N$? Compare this with $\mathrm{Var}(\bar{y}_N)$.

d. What is the asymptotic standard deviation of $\bar{y}_N$?

e. How would you obtain the asymptotic standard error of $\bar{y}_N$?

Solution

a. Because $\mathrm{Var}(\bar{y}_N) = \sigma^2/N$,

$$\mathrm{Var}[\sqrt{N}(\bar{y}_N - \mu)] = N(\sigma^2/N) = \sigma^2.$$

b. By the CLT,

$$\sqrt{N}(\bar{y}_N - \mu) \xrightarrow{d} \mathrm{Normal}(0, \sigma^2),$$

and so

$$\mathrm{Avar}[\sqrt{N}(\bar{y}_N - \mu)] = \sigma^2.$$

c. We obtain $\mathrm{Avar}(\bar{y}_N)$ by dividing $\mathrm{Avar}[\sqrt{N}(\bar{y}_N - \mu)]$ by N. Therefore,

$$\mathrm{Avar}(\bar{y}_N) = \sigma^2/N.$$

As expected, this coincides with the actual variance of $\bar{y}_N$.

d. The asymptotic standard deviation of $\bar{y}_N$ is the square root of its asymptotic variance, or

$$\sigma/\sqrt{N}.$$

e. To obtain the asymptotic standard error of $\bar{y}_N$, we need a consistent estimator of σ. Typically, the unbiased estimator of σ^2 is used:

$$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y}_N)^2,$$

and then $\hat{\sigma}$ is the positive square root. The asymptotic standard error of $\bar{y}_N$ is simply

$$\hat{\sigma}/\sqrt{N}.$$

Problem 9

Let Y be a binomial random variable with parameters $n = 100$ and $p = \frac{1}{2}$.

Using the Central Limit Theorem, show that a normal random variable X with mean $\mu = 50$ and variance $\sigma^2 = 25$ can be used to approximate Y.

Solution

A binomial random variable Y with $n = 100$, $p = \frac{1}{2}$ can be written as:

$$Y = \sum_{i=1}^{100} X_i$$

where $X_1, \ldots, X_{100}$ are i.i.d. Bernoulli with $p = \frac{1}{2}$. Thus:

$$Y = 100 \cdot \left(\frac{1}{100} \sum_{i=1}^{100} X_i \right) = 100 \cdot \overline{X}_{100}$$

From the CLT, we know:

$$\overline{X}_{100} \sim N\left(\frac{1}{2}, \frac{1}{400}\right)$$

Therefore:

$$Y \sim N\left(\frac{1}{2} \cdot 100,\ \frac{1}{400} \cdot 100^2\right) = N(50,\ 25)$$

So Y can be approximated by a normal distribution with mean $\mu = 50$ and variance $\sigma^2 = 25$.

Problem 10

Let X be an integrable[1] random variable defined on a sample space Ω. Let X be a positive random variable[2]. Let $c \in \mathbb{R}_{++}$. Prove the following inequality, called **Markov's inequality**:

$$\mathbb{P}(X \geq c) \leq \frac{\mathbb{E}[X]}{c}$$

Solution

First note that

$$\mathbf{1}_{\{X \geq c\}} + \mathbf{1}_{\{X < c\}} = 1$$

where $\mathbf{1}_{\{X \geq c\}}$ is the indicator[3] of the event $\{X \geq c\}$, and $\mathbf{1}_{\{X < c\}}$ is the indicator of the event $\{X < c\}$. As a consequence, we can write

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X \cdot 1] \\
&= \mathbb{E}\left[X \cdot (\mathbf{1}_{\{X \geq c\}} + \mathbf{1}_{\{X < c\}})\right] \\
&= \mathbb{E}[X\mathbf{1}_{\{X \geq c\}}] + \mathbb{E}[X\mathbf{1}_{\{X < c\}}]
\end{aligned}$$

Now, note that $X\mathbf{1}_{\{X < c\}}$ is a positive random variable and that the expected value of a positive random variable is positive[4]:

$$\mathbb{E}[X\mathbf{1}_{\{X < c\}}] \geq 0$$

Therefore,

[1] That is, $\mathbb{E}[|X|] < \infty$.
[2] That is, $X(\omega) \geq 0$ for all $\omega \in \Omega$.
[3] That is, a function that is 1 when the event is true, and 0 otherwise.
[4] This follows from the definition of the expectation for nonnegative random variables.

$$\mathbb{E}[X] = \mathbb{E}[X\mathbf{1}_{\{X \geq c\}}] + \mathbb{E}[X\mathbf{1}_{\{X < c\}}] \geq \mathbb{E}[X\mathbf{1}_{\{X \geq c\}}]$$

The random variable $c \cdot \mathbf{1}_{\{X \geq c\}}$ is less than or equal to the random variable $X \cdot \mathbf{1}_{\{X \geq c\}}$ for any $\omega \in \Omega$:

$$c \cdot \mathbf{1}_{\{X \geq c\}} \leq X \cdot \mathbf{1}_{\{X \geq c\}}$$

because c is always smaller than X when the indicator $\mathbf{1}_{\{X \geq c\}}$ is not zero. Since the expected value operator preserves inequalities[5], we have

$$c \cdot \mathbf{1}_{\{X \geq c\}} \leq X \cdot \mathbf{1}_{\{X \geq c\}} \Longrightarrow \mathbb{E}[c \cdot \mathbf{1}_{\{X \geq c\}}] \leq \mathbb{E}[X \cdot \mathbf{1}_{\{X \geq c\}}]$$

Furthermore, by using the linearity of the expected value[6] and the fact that the expected value of an indicator is equal to the probability of the event it indicates[7], we obtain

$$\mathbb{E}[c \cdot \mathbf{1}_{\{X \geq c\}}] = c\mathbb{E}[\mathbf{1}_{\{X \geq c\}}] = c\mathbb{P}(X \geq c) \Longrightarrow c\mathbb{P}(X \geq c) \leq \mathbb{E}[X\mathbf{1}_{\{X \geq c\}}]$$

The above inequalities can be put together:

$$\mathbb{E}[X] \geq \mathbb{E}[X\mathbf{1}_{\{X \geq c\}}]\mathbb{E}[X\mathbf{1}_{\{X \geq c\}}] \geq c\mathbb{P}(X \geq c) \Longrightarrow \mathbb{E}[X] \geq c\mathbb{P}(X \geq c)$$

Finally, since c is strictly positive, we can divide both sides of the right-hand inequality to obtain Markov's inequality:

$$\mathbb{P}(X \geq c) \leq \frac{\mathbb{E}[X]}{c}$$

■

Problem 11

Let X be a square integrable[8] random variable defined on a sample space Ω. Let μ and σ^2 denote the mean and variance of X respectively. Let $k \in \mathbb{R}_{++}$. Prove the following inequality, called **Chebyshev's inequality**:

$$\mathbb{P}(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

[5] This is a fundamental property of the expectation operator.
[6] That is, $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ for random variables X, Y and scalars a, b.
[7] That is, $\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A)$.
[8] That is, $\mathbb{E}[(X - \mu)^2] < \infty$.

Solution

The proof is a straightforward application of Markov's inequality (29.1). Since $(X-\mu)^2$ is a positive random variable, we can apply Markov's inequality with $c = k^2$ to obtain

$$\mathbb{P}\left((X-\mu)^2 \geq k^2\right) \leq \frac{\mathbb{E}[(X-\mu)^2]}{k^2}$$

But $(X-\mu)^2 \geq k^2$ if and only if $|X-\mu| \geq k$; so, we can write

$$\mathbb{P}(|X-\mu| \geq k) \leq \frac{\mathbb{E}[(X-\mu)^2]}{k^2}$$

Furthermore, by the very definition of variance, we have

$$\mathbb{E}[(X-\mu)^2] = \mathrm{Var}[X] = \sigma^2$$

Therefore,

$$\mathbb{P}(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

■

Problem 12

Let X be an integrable random variable. Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function such that

$$Y = g(X)$$

is also integrable. Prove the following inequality, called **Jensen's inequality**:

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$$

Solution

A function g is convex if, for any point x_0, the graph of g lies entirely above its tangent at the point x_0:

$$g(x) \geq g(x_0) + b(x - x_0), \quad \forall x$$

where b is the slope of the tangent. By setting $x = X$ and $x_0 = \mathbb{E}[X]$, the inequality becomes

$$g(X) \geq g(\mathbb{E}[X]) + b(X - \mathbb{E}[X])$$

By taking the expected value of both sides of the inequality, and by using the fact that the expected value operator preserves inequalities[9], we obtain

$$\begin{aligned}
\mathbb{E}[g(X)] &\geq \mathbb{E}\left[g(\mathbb{E}[X]) + b(X - \mathbb{E}[X])\right] \\
\boxed{\text{A}} \quad &= g(\mathbb{E}[X]) + b(\mathbb{E}[X] - \mathbb{E}[X]) \\
&= g(\mathbb{E}[X])
\end{aligned}$$

where: in step $\boxed{\text{A}}$ we have used the linearity of the expected value. ■

[9]That is, if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.