International School of Economics at TSU
Econometrics II
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Problem Set 3

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1: The data $\{y_i, x_i, w_i\}$ is from a random sample, $i = 1, \dots, n$. The parameter β is estimated by minimizing the criterion function:

$$S(\beta) = \sum_{i=1}^{n} w_i (y_i - x_i'\beta)^2$$

That is,

$$\hat{\beta} = \arg\min_{\beta} S(\beta)$$

(a) Find an explicit expression for $\hat{\beta}$.

(b) What population parameter is $\hat{\beta}$ estimating?

Be explicit about any assumptions you need to impose, but don't make more assumptions than necessary.

(c) Find the probability limit for $\hat{\beta}$ as $n \to \infty$.

(d) Find the asymptotic distribution of $\sqrt{n}(\hat{\beta} - \beta)$ as $n \to \infty$.

Solution:

(a) Using matrix notation

$$\hat{\beta} = (X'WX)^{-1}X'WY$$

where $W = \text{diag}(w_1, \dots, w_n)$. Alternatively

$$\hat{\beta} = \left(\sum_{i=1}^{n} w_i x_i x_i' \right)^{-1} \sum_{i=1}^{n} w_i x_i y_i$$

(b) What parameter is $\hat{\beta}$ estimating?

It appears to be estimating

$$\beta = \left(\mathbb{E}(w_i x_i x_i')\right)^{-1} \mathbb{E}(w_i x_i y_i),$$

a weighted projection. This imposes no assumptions beyond the existence of moments and the invertibility of $\mathbb{E}(w_i x_i x_i')$.

As an alternative, you might state that $\hat{\beta}$ is estimating the slope parameter in the regression model:

$$\begin{aligned} y_i &= x_i'\beta + e_i \\ \mathbb{E}(e_i \mid x_i, w_i) &= 0 \end{aligned}$$

but this is more restrictive than the simple weighted projection.

(c) Probability limit of $\hat{\beta}$

By the WLLN,

$$\frac{1}{n} \sum_{i=1}^{n} w_i x_i x_i' \xrightarrow{p} \mathbb{E}(w_i x_i x_i'), \quad \frac{1}{n} \sum_{i=1}^{n} w_i x_i y_i \xrightarrow{p} \mathbb{E}(w_i x_i y_i)$$

By the CMT,

$$\hat{\beta} = \left(\frac{1}{n} \sum_{i=1}^{n} w_i x_i x_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} w_i x_i y_i \right) \xrightarrow{p} \beta$$

as defined in part (b).

(d) Asymptotic distribution

With β defined in part (b), we then **define** the error by the equation:

$$y_i = x_i'\beta + e_i$$

It is important that e_i be defined, as it is not given in the question!
It is also important that the definition be consistent with your answer in part (b). Then

$$\hat{\beta} = \left(\sum_{i=1}^{n} w_i x_i x_i'\right)^{-1} \sum_{i=1}^{n} w_i x_i y_i = \beta + \left(\sum_{i=1}^{n} w_i x_i x_i'\right)^{-1} \sum_{i=1}^{n} w_i x_i e_i$$

so

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}\sum_{i=1}^{n} w_i x_i x_i'\right)^{-1} \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i x_i e_i\right)$$

As $n \to \infty$,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i x_i e_i \xrightarrow{d} \mathcal{N}(0, \Omega)$$

where

$$\Omega = \mathbb{E}(w_i^2 x_i x_i' e_i^2)$$

Then letting $Q = \mathbb{E}(w_i x_i x_i')$, we have

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, Q^{-1}\Omega Q^{-1})$$

Problem 2: Take the linear equation $Y = Z\beta + e$, and consider the following estimators of β:

(a) $\hat{\beta}_1$: 2SLS using the instruments X_1

(b) $\hat{\beta}_2$: 2SLS using the instruments X_2

(c) $\tilde{\beta}$: GMM using the instruments $X = (X_1, X_2)$ and the weight matrix

$$W = \begin{pmatrix} (X_1'X_1)^{-1}\lambda & 0 \\ 0 & (X_2'X_2)^{-1}(1-\lambda) \end{pmatrix}$$

for $\lambda \in (0, 1)$.

Find an expression for $\tilde{\beta}$ which shows that it is a specific weighted average of $\hat{\beta}_1$ and $\hat{\beta}_2$.

Solution:

Let

$$\begin{aligned} D_1 &= Z'P_1Z \\ D_2 &= Z'P_2Z \\ D_\lambda &= \lambda D_1 + (1-\lambda) D_2 \end{aligned}$$

Recall that

$$\begin{aligned} \hat{\beta}_1 &= D_1^{-1} Z'P_1Y \\ P_1 &= X_1 (X_1'X_1)^{-1} X_1' \\ \hat{\beta}_2 &= D_2^{-1} Z'P_2Y \\ P_2 &= X_2 (X_2'X_2)^{-1} X_2' \end{aligned}$$

and we calculate that

and we calculate that

$$\begin{aligned} \tilde{\beta} &= (Z'XWX'Z)^{-1} (Z'XWX'Y) \\ &= \left(\begin{pmatrix} Z'X_1 & Z'X_2 \end{pmatrix} \begin{pmatrix} (X_1'X_1)^{-1}\lambda & 0 \\ 0 & (X_2'X_2)^{-1}(1-\lambda) \end{pmatrix} \begin{pmatrix} X_1'Z \\ X_2'Z \end{pmatrix} \right)^{-1} \\ &\quad \cdot \left(\begin{pmatrix} Z'X_1 & Z'X_2 \end{pmatrix} \begin{pmatrix} (X_1'X_1)^{-1}\lambda & 0 \\ 0 & (X_2'X_2)^{-1}(1-\lambda) \end{pmatrix} \begin{pmatrix} X_1'Y \\ X_2'Y \end{pmatrix} \right) \\ &= (\lambda Z'P_1Z + (1-\lambda) Z'P_2Z)^{-1} (\lambda Z'P_1Y + (1-\lambda) Z'P_2Y) \\ &= D_\lambda^{-1} \lambda Z'P_1Y + D_\lambda^{-1} (1-\lambda) Z'P_2Y \\ &= \lambda D_\lambda^{-1} D_1 \hat{\beta}_1 + (1-\lambda) D_\lambda^{-1} D_2 \hat{\beta}_2 \\ &= W_1 \hat{\beta}_1 + W_2 \hat{\beta}_2 \end{aligned}$$

where $W_1 = \lambda D_\lambda^{-1} D_1$ and $W_2 = (1-\lambda) D_\lambda^{-1} D_2$. $\tilde{\beta}$ is a weighted average since

$$W_1 + W_2 = D_\lambda^{-1} (\lambda D_1 + (1-\lambda) D_2) = I$$

Problem 3: The observations are iid, $(y_i, x_i, q_i : i = 1, \ldots, n)$, where x_i is $k \times 1$ and q_i is $m \times 1$. The model is

$$
\begin{aligned}
y_i &= x_i'\beta + e_i \\
\mathbb{E}(x_i e_i) &= 0 \\
\mathbb{E}(q_i e_i) &= 0
\end{aligned}
$$

Find the efficient GMM estimator for β.

Solution:

The efficient GMM estimator is

$$
\begin{aligned}
\hat{\beta} &= \left(X' \begin{pmatrix} X & Q \end{pmatrix} \hat{\Omega}^{-1} \begin{pmatrix} X' \\ Q' \end{pmatrix} X \right)^{-1} \left(X' \begin{pmatrix} X & Q \end{pmatrix} \hat{\Omega}^{-1} \begin{pmatrix} X' \\ Q' \end{pmatrix} Y \right) \\
&= \left(\begin{pmatrix} X'X & X'Q \end{pmatrix} \hat{\Omega}^{-1} \begin{pmatrix} X'X \\ Q'X \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} X'X & X'Q \end{pmatrix} \hat{\Omega}^{-1} \begin{pmatrix} X'Y \\ Q'Y \end{pmatrix} \right)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\Omega} &= \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} x_i \\ q_i \end{pmatrix} \begin{pmatrix} x_i' & q_i' \end{pmatrix} \hat{e}_i^2 \\
&= \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} x_i x_i' & x_i q_i' \\ q_i x_i' & q_i q_i' \end{pmatrix} \hat{e}_i^2
\end{aligned}
$$

and $\hat{e}_i = y_i - x_i'\tilde{\beta}$ with $\tilde{\beta}$ a preliminary consistent estimator. One possibility is $\tilde{\beta} = (X'X)^{-1}(X'Y)$, the LS estimator.

Problem 4: The model is

$$
\begin{aligned}
y_i &= z_i \beta + x_i \gamma + e_i \\
\mathbb{E}(e_i \mid x_i) &= 0
\end{aligned}
$$

Thus z_i is potentially endogenous and x_i is exogenous. Assume that $z_i \in \mathbb{R}$ and $x_i \in \mathbb{R}$. Someone suggests estimating (β, γ) by GMM, using the pair (x_i, x_i^2) as the instruments. Is this feasible? Under what conditions, if any, (in addition to those described above) is this a valid estimator?

Solution:

This is feasible since there are two parameters to estimate (β and γ) and two instruments (x_i and x_i^2). This is the just-identified case. Under the given assumtpion $E(e_i \mid x_i) = 0$, we know that $E(x_i e_i) = 0$ and $E\left(x_i^2 e_i\right) = 0$, so these are valid instrumental variables. For identification, we need that the included endogenous variable (z_i) be correlated with the excluded exogenous variabe (x_i^2) after controlling for the included exogenous variable (x_i). That is, in the reduced form regression

$$z_i = x_i \alpha_1 + x_i^2 \alpha_2 + u_i \tag{1}$$

it must be that $\alpha_2 \neq 0$. Thus the conditional mean of z_i given x_i cannot be linear, it must be a non-linear relationship. Note: This is not the same thing $E\left(z_i x_i^2\right) \neq 0$, but it is close.

In summary, the proposed GMM estimator is valid, if coefficient α_2 in the reduced form equation (1) is non-zero.

Problem 5: Consider the model

$$y_i = x_i' \beta + e_i$$
$$\mathbb{E}(e_i \mid z_i) = 0$$

with y_i scalar and x_i and z_i each a k vector. You have a random sample $(y_i, x_i, z_i : i = 1, \dots, n)$.

(a) Write the IV estimator $\hat{\beta}$ for β

(b) Suppose that x_i is exogenous in the sense that $\mathbb{E}(e_i \mid z_i, x_i) = 0$. Is $\hat{\beta}$ unbiased for β?

(c) Continuing to assume that x_i is exogenous, find the variance matrix for $\hat{\beta}$, $\text{var}(\hat{\beta} \mid X, Z)$.

Solution:

(a) $\widehat{\beta} = (Z'X)^{-1} Z'Y = \left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i y_i\right)$

(b) Write

$$\widehat{\beta} - \beta = \left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i e_i\right)$$

Then

$$\begin{aligned} E\left(\widehat{\beta} - \beta \mid Z, X\right) &= E\left(\left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i e_i\right) \mid Z, X\right) \\ &= \left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i E\left(e_i \mid Z, X\right)\right) \\ &= \left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i E\left(e_i \mid z_i, x_i\right)\right) \\ &= 0 \end{aligned}$$

Thus by the law of iterated expectations

$$E\left(\widehat{\beta}\right) = \beta$$

and $\widehat{\beta}$ is unbiased for β.

(c) Since $E\left(\widehat{\beta} \mid Z, X\right) = \beta$

$$\begin{aligned} \operatorname{var}\left(\widehat{\beta} | X, Z\right) &= E\left(\left(\widehat{\beta} - \beta\right)\left(\widehat{\beta} - \beta\right)' \mid Z, X\right) \\ &= E\left(\left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i e_i\right)\left(\sum_{i=1}^{n} e_i z_i'\right)\left(\sum_{i=1}^{n} x_i z_i'\right)^{-1} \mid Z, X\right) \\ &= \left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} E\left(z_i e_i e_j z_j' \mid Z, X\right)\right)\left(\sum_{i=1}^{n} x_i z_i'\right)^{-1} \\ &= \left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} E\left(z_i e_i e_i z_i' \mid z_i, x_i\right)\right)\left(\sum_{i=1}^{n} x_i z_i'\right)^{-1} \\ &= \left(\sum_{i=1}^{n} z_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i z_i' \sigma_i^2\right)\left(\sum_{i=1}^{n} x_i z_i'\right)^{-1} \end{aligned}$$

where

$$\sigma_i^2 = E\left(e_i^2 \mid z_i, x_i\right)$$