Microeconomics IV (Game Theory)

Lecture 2 – Basic Models of Game Theory

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Last Week

The Big Picture

- **Lecture 1 Goal:** show that any rational, intelligent agent behaves as if maximizing expected utility

- The argument has three steps:

Step	Content
1	Define the **object of choice**: lotteries $f : \Omega \to \Delta(X)$
2	Impose **8 consistency axioms** on preferences over lotteries
3	Show preferences satisfying the axioms $\iff$ **SEU maximization**

- Two outputs of the theorem: a **utility function** $u(x, t)$ and **subjective beliefs** $p(t \mid S)$
- Both are elicited from observable preferences – not assumed

Lotteries – The Object of Choice

- X = finite set of **prizes**; Ω = finite set of **states of the world**
- Two sources of uncertainty:
  - **Risk (objective):** probabilities known – coin tosses, roulette wheels
  - **Uncertainty (subjective):** probabilities unknown – competitor's strategy, market demand

Definition: Lottery

A **lottery** is any function $f : \Omega \to \Delta(X)$, where $f(x \mid t)$ is the objective probability of prize x given state t.

- Nature picks state t (subjective uncertainty); then $f(\cdot \mid t)$ determines the prize by objective randomization
- The agent chooses f **before** either is resolved
- This compound structure lets the theorem separately identify $u(x, t)$ and $p(t \mid S)$

The Eight Axioms

Axiom	Name	What it requires
1A	Completeness	Any two lotteries can be ranked
1B	Transitivity	Rankings are consistent across comparisons
2	Relevance	Only outcomes inside S matter given S
3	Monotonicity	More probability on the better option is better
4	Continuity	No lottery is infinitely better than another
5A/B	Objective Substitution	Mixing with the same lottery preserves rankings
6A/B	Subjective Substitution	Prefer f over g in S and in $T \Rightarrow$ prefer f in $S \cup T$
7	Interest	Something is at stake in every state
8	State Neutrality	(Optional) Same prize valued equally across states

- Axioms 5 and 6 are the deepest – violated by the **Allais paradox** (Axiom 5B) and the **Ellsberg paradox** (Axiom 6)

Subjective Expected Utility

Subjective Expected Utility

Axioms 1AB, 2, 3, 4, 5AB, 6AB, 7 are jointly satisfied **if and only if** $\exists$ utility $u : X \times \Omega \to \mathbb{R}$ and conditional probabilities $p : \Xi \to \Delta(\Omega)$ such that:

$$\max_x u(x,t) = 1, \quad \min_x u(x,t) = 0 \qquad \forall t \in \Omega \tag{1}$$

$$p(R \mid T) = p(R \mid S) \cdot p(S \mid T) \qquad \forall R \subseteq S \subseteq T \subseteq \Omega \tag{2}$$

$$f \succsim_S g \iff \sum_{t \in S} p(t \mid S) \sum_{x \in X} u(x,t)\, f(x \mid t) \;\geq\; \sum_{t \in S} p(t \mid S) \sum_{x \in X} u(x,t)\, g(x \mid t) \tag{3}$$

Adding Axiom 8 $\iff$ (1)–(3) hold with state-independent $u(x,t) = U(x)$.

- (1): utility normalized to $[0,1]$ in every state
- (2): beliefs satisfy Bayes's formula (chain rule)
- (3): agent always chooses the lottery with higher **expected utility**

Introduction

Modeling Games

- Every game begins with a **model** that describes the situation
- A model must not be too simple or too complex:
  - Too simple: misses key features of the game
  - Too complex: hides the essential logic
- The goal is to capture the *minimal* structure needed to identify the strategic problem

Two Forms of Representation

- Two key ways to model games:
  - **Extensive form**: detailed tree of actions and information
  - **Strategic (normal) form**: simplified table of payoffs
- Extensive form is richer and captures timing and information explicitly
- Strategic form abstracts away timing – it is a planning-stage representation

Note

Both forms are exact – neither loses information about rational behavior, under the right conditions.

Extensive-Form Games

Origins of the Extensive Form

- Standard definition of extensive form is due to Kuhn (1953)
- Built on von Neumann and Morgenstern (1944)
- Strategic and Bayesian forms are simpler but **derived** from the extensive form
    - Expected payoffs are computed by summing over terminal nodes of the tree

A Simple Card Game

- Two players: *Player 1* and *Player 2*
- Each contributes $1 to a pot
- Player 1 draws a card (red or black) and decides:
  - **Raise** (add $1 more), or
  - **Fold** (end the game)
- If Player 1 raises, Player 2 decides:
  - **Meet** (add $1 more), or
  - **Pass**

Game Outcomes

- If Player 1 **folds**:
    - Player 1 wins if red, loses if black
- If Player 2 **passes**:
    - Player 1 wins
- If Player 2 **meets**:
    - Player 1 shows the card
    - Red: Player 1 wins, gets $2
    - Black: Player 2 wins, gets $2

Tree Representation

- The game can be shown as a **tree diagram**:
  - **Nodes**: decision points
  - **Branches**: possible actions
  - **Root**: beginning of the game
  - **Terminal nodes**: outcomes with payoffs

- Payoffs shown as pairs (x_1, x_2) for Players 1 and 2
- Example: $(2, -2)$ means Player 1 wins 2, Player 2 loses 2

Tree Diagram Example

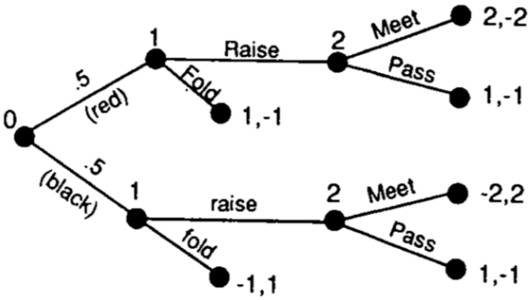


Figure 1: Game 1

- Payoffs shown as pairs (x_1, x_2) for Players 1 and 2
- For example:
  - $(2, -2)$ means Player 1 wins 2, Player 2 loses 2

Chance and Decision Nodes

- Nodes labeled “0” are **chance nodes** (e.g. drawing a card)
- Others (e.g. “1”, “2”) are **decision nodes** controlled by players
- Probability of red or black is 0.5 each

Incomplete Information

- Player 1 sees the card color; Player 2 does not
- In the tree, Player 2 must act without knowing the color
- She must choose the same action at both of her nodes

Note

This is the key departure from perfect-information games: a player may not know exactly where in the tree she is when she moves.

Information Sets – The Core Idea

- When a player cannot distinguish between two nodes, we say they belong to the **same information set**
- An information set is a group of nodes that are **indistinguishable** to the player at the time of the move
- The player must choose a single action that applies to all nodes in the set

- **Why does this matter?** It is what forces a player to act the same way in situations she cannot tell apart
- The information set is the game-theoretic formalization of *what a player knows*

Information Sets

- We label nodes with:
  - **Player number**
  - **Information state**
- Example: “2.0” means Player 2, in information state 0
- Information state indicates what a player knows at that node

Improved Tree with Information

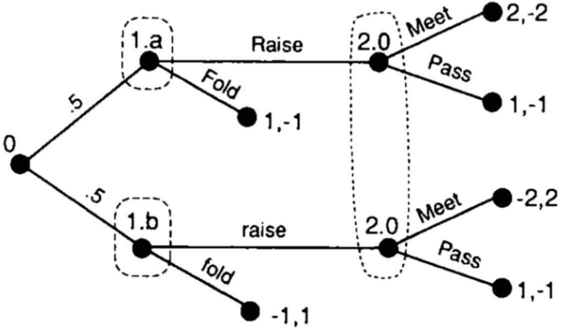


Figure 2: Game 2

- Dashed lines show **information sets**
- Player 2 cannot distinguish between red and black branches

Move Labels and Player Knowledge

- We keep branch labels for player decisions, but not for chance events
- Move labels matter because players choose actions, not specific branches
- A player must know all her options at any decision point

Information Sets and Choices

- Player 2 must make the same decision at both of her nodes
- She does not know if Player 1 drew red or black
- So both of her nodes must have the same set of move labels: "Meet" and "Pass"

Ensuring Meaningful Play

- If we added a third branch only after one of Player 2's nodes, she could exploit hidden information
- To be valid, both nodes must offer the same moves if in the same information set

Note

The consistency requirement on move labels is not just technical – it encodes the idea that a player cannot act on information she does not have.

Player 1's Information

- Player 1 knows the card's color
- So he can have different move labels at different nodes
- Player 2 cannot, because she cannot tell nodes apart

Importance of Move Labels

- Move labels shape the player's choices
- In our game, "Pass" is better for Player 2 if Player 1 has red
- "Meet" is better if Player 1 has black
- But Player 2 does not know the color when choosing

What is a Path?

- A **path** is a sequence of connected branches in a graph
- Formally, a path is:

$$\{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{m-1}, x_m\}\} = \{\{x_k, x_{k+1}\}\} \quad \text{for } k = 1, \dots, m-1$$

- Here:
  - $m \geq 2$
  - Each x_k is a different node

Basic Graph Terminology

- A **graph** is a set of:
  - **Nodes** (points)
  - **Branches** (connections between nodes)
- A **branch** connects two nodes

What is a Tree?

- A **tree** is a graph with a special structure:
  - Between any two nodes, there is **exactly one** path
- This makes it easy to trace the flow of a game or process

Rooted Tree

- A **rooted tree** has a designated starting node called the **root**
- In diagrams, the root is usually on the **left**
- All play begins at the root

Paths in Rooted Trees

- The **path to a node** is the **unique sequence of branches** connecting it to the root
- This structure lets us track the history of the game step by step

Alternatives and Branching

- An **alternative** is any branch **not** on the path to a node
- For example, if the current path is $\{x_1, x_2, x_3\}$, then a branch from x_2 to another node is an alternative

Follows and Immediate Follows

- A branch or node x **follows** y if y is **in the path** to x
- x **immediately follows** y if:
    - x directly follows y, and
    - There is a branch $\{x, y\}$

Terminal Nodes

- A **terminal node** is a node with **no further branches**
- It represents an **end point** in the game
- At terminal nodes, we specify the **payoffs** for players

Definition: n-Person Extensive-Form Game (1/3)

An n-person extensive-form game Γ^e consists of a rooted tree with labeled nodes and branches. The following conditions must be satisfied:

1. **Player Labels** Each nonterminal node is labeled with a number in $\{0, 1, \dots, n\}$
   - Label 0: **chance node**, controlled by nature
   - Label $i \in \{1, \dots, n\}$: **decision node**, controlled by player i
2. **Chance Probabilities** Every branch from a chance node has a nonnegative probability label summing to 1:

$$\sum_{j=1}^{k} p_j = 1, \quad p_j \geq 0$$

Definition: n-Person Extensive-Form Game (2/3)

3. **Information States** Every node controlled by a player is assigned an **information state**
   - A player knows only the information state at the time of the move
   - Nodes are indistinguishable if the player cannot tell which was reached

   Notation: node labeled $i.k$ means player i, information state k

4. **Move Labels** Each alternative (branch) at a node has a **move label**
   - If two nodes share the same player and information label, the set of move labels must be **identical**
   - This ensures players know their available actions even without knowing the exact node

Definition: n-Person Extensive-Form Game (3/3)

5. **Terminal Node Payoffs** Each terminal node is labeled with a vector of utilities:

$$(u_1, u_2, \dots, u_n)$$

   - u_i: utility (payoff) to player i

6. **Perfect Recall**

   Players remember everything they knew earlier:

   - All previous moves they made
   - All past information states they experienced
   - If nodes y and z are indistinguishable and a decision was made at y, a matching indistinguishable move must exist at z
   - This avoids contradictions in player memory and ensures coherent strategy

Perfect Information and Strategies (1/2)

- A game has **perfect information** if:
  - No two nodes share the same information state
  - The player always knows:
    - All previous moves by others
    - Their own past moves
    - What node they are at
- In this case, the player knows exactly what has happened before making any move

Perfect Information and Strategies (2/2)

Strategy in Extensive-Form Games

A **strategy** is a complete plan specifying what a player will do at **every** information state – including states that may never be reached in actual play. Formally, let S_i be the set of information states for player i and D_s the set of moves at state s. The set of strategies for player i is:

$$\prod_{s \in S_i} D_s$$

Why Must a Strategy Cover Unreachable States?

- A strategy is a **plan made before the game begins** – a complete contingent decision rule
- Even if an information state is never reached, the player must decide *in advance* what they would do there

- **Why?** Because other players' decisions depend on their *beliefs* about what you would do at every node
  - Player 2 chooses whether to meet partly based on what would happen if she does
  - That belief is only well-defined if Player 1 has a plan at every information state

- **Analogy**: a chess player preparing for a tournament writes down what they would do in every position – including positions they may never reach – because the opponent's strategy depends on it

Why Must a Strategy Cover Unreachable States? (Cont.)

- A second reason: **consistency in equilibrium analysis**
  - When we check whether a strategy profile is an equilibrium, we ask: does any player want to deviate?
  - To answer this, we must evaluate payoffs from deviating – which requires knowing what happens at information states that are **off the equilibrium path**
  - Without a complete plan, we cannot check whether a player prefers to deviate

Note

This is why "I would never reach that node anyway" is not a valid reason to leave a strategy undefined. The definition of equilibrium forces us to evaluate the full plan.

Example: Strategies in the Card Game

- Player 1 has 2 information states:
  - $1.a$: sees red card
  - $1.b$: sees black card
- Each strategy is a pair of choices at $1.a$ and $1.b$:
  - {Rr, Rf, Fr, Ff}

Note

- First letter: move at $1.a$ (capital for red card)
- Second letter: move at $1.b$ (lowercase for black card)
- Rf = "Raise if red, Fold if black"
- Rr = "Raise regardless of card color"

Player 2's Strategy

- Player 2 has only 1 information state
- So she has just 2 strategies:
  - M = "Meet"
  - P = "Pass"
- Even though only one path is realized, a strategy covers **every possible contingency**

Examples

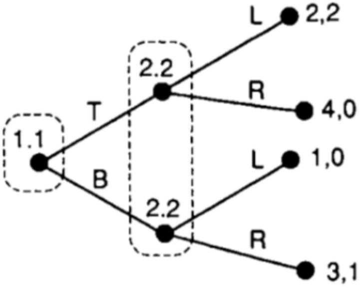


Figure 3: Game 3

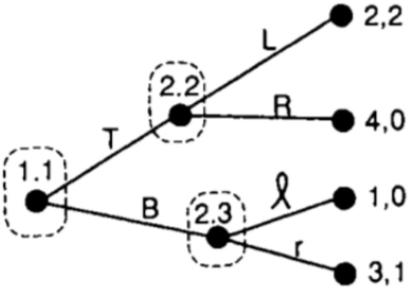


Figure 4: Game 4

Game 3 – Limited Information

- Player 2 must choose between **L** and **R**
- She **does not observe** Player 1's move (T or B)
- Player 1 anticipates this and will choose **T**, since:
  - Against either L or R, T gives a better payoff than B
- Example outcome:
  - If Player 1 chooses T, and Player 2 chooses L, Player 2 gets payoff 2

Game 4 – Full Information

- Player 2's two decision nodes now have **different information labels**
- She observes whether Player 1 chose T or B
- This changes Player 2's strategy set to:

$$\{\mathrm{L}\ell, \mathrm{L}r, \mathrm{R}\ell, \mathrm{R}r\}$$

- Player 2 can now condition her action on Player 1's move

Strategic Implications

- In Game 3:
  - Player 2's strategies: $\{\text{L}, \text{R}\}$
- In Game 4:
  - Player 2's strategies: $\{\text{L}\ell, \text{L}r, \text{R}\ell, \text{R}r\}$ (First letter $=$ move if T, second $=$ move if B)
- If Player 2 is rational:
  - She will follow **Lr**: choose L if T, r if B
  - Player 1 anticipates this and chooses **B**, getting payoff 3

Strategic Reasoning in Game 4

- Suppose Player 1 plays B
- Player 2 will respond with **r** in both:
  - Strategy Rr (“R if T, r if B”)
  - Strategy Lr (“L if T, r if B”)
- Why should Player 1 choose B?
  - Because Player 2 prefers L if T and r if B ⇒ strategy Lr
  - Then Player 1 expects payoff:
    - 2 from T
    - 3 from B ⇒ **chooses B**

Information Structure and Equilibrium

- The change from Game 3 to Game 4 increases Player 2's strategy space
- This can:
  - Change Player 1's optimal move
  - Lower Player 2's expected payoff (despite having more info)
- If Player 2 used strategy Rr:
  - Player 1 would choose T and get payoff 4

Note

More information does not always help – it can change the equilibrium in ways that harm the better-informed player.

Strategic Form

Strategic Form Representation

- The **strategic form** (or normal form) is a simpler way to represent a game
- It specifies:
  - A set of players N
  - Each player's strategy set C_i
  - A utility function u_i for each player

$$\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$$

- $u_i : \prod_{j \in N} C_j \to \mathbb{R}$ gives player i's payoff from any strategy profile

Strategy Profiles and Timing

- A **strategy profile** $c = (c_1, \dots, c_n)$ is a combination of strategies, one for each player
- All players choose their strategies **simultaneously**
- The strategic form is a **static model**:
    - It abstracts from the timing of moves
    - It only captures the strategic planning stage

From Extensive to Strategic Form

- The strategic form can be derived from an extensive-form game
- Example: the card game in Game 2
  - Player 1's strategy set:
  $$C_1 = \{\mathrm{Rr}, \mathrm{Rf}, \mathrm{Fr}, \mathrm{Ff}\}$$
  - Player 2's strategy set:
  $$C_2 = \{\mathrm{M}, \mathrm{P}\}$$
- Players choose these strategies **before the card is revealed**

Example: Strategy Profile (Rf, M)

- Strategy Rf: Raise if red, Fold if black
- Strategy M: Player 2 meets if Player 1 raises
    - Expected payoffs:

$$u_1(\text{Rf}, \text{M}) = 2 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0.5$$

$$u_2(\text{Rf}, \text{M}) = -2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = -0.5$$

- These are **expected utilities** given uncertainty about the card draw

Game 2 in Normal Form (1/2)

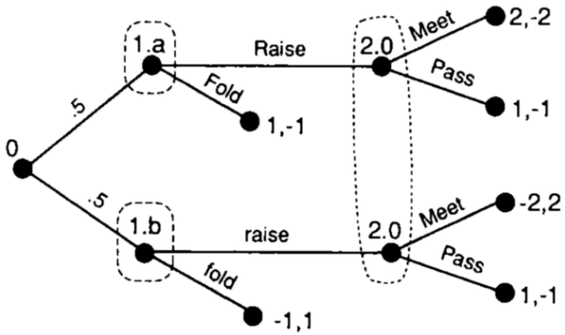


Figure 5: Game 2

Game 2 in Normal Form (2/2)

	C_2	
C_1	M	P
Rr	0,0	1,−1
Rf	0.5,−0.5	0,0
Fr	−0.5,0.5	1,−1
Ff	0,0	0,0

Figure 6: Game 2 in a Normal Form

Mapping Extensive Form to Strategic Form (1/3)

- Let Γ^e be an extensive-form game
- To construct its strategic form Γ, define:
  - N: set of players
  - C_i: strategy set for player i
  - $c_i \in C_i$: function assigning a move $c_i(r)$ at each information state r
  - $c = (c_1, \ldots, c_n)$: a **strategy profile**

Mapping Extensive Form to Strategic Form (2/3)

- We define $P(x \mid c)$ as the probability that node x is reached given strategy profile c
- Rules:
  - If x is the root: $P(x \mid c) = 1$
  - If x follows a chance node y:

$$P(x \mid c) = q \cdot P(y \mid c) \quad \text{where } q \text{ is the chance probability}$$

  - If x follows a decision node y for player i in state r:
    - $P(x \mid c) = P(y \mid c)$ if $c_i(r)$ selects move from y to x
    - $P(x \mid c) = 0$ otherwise

Mapping Extensive Form to Strategic Form (3/3)

- Let Ω^* be the set of all **terminal nodes** of Γ^e
- Let $w_i(x)$ be the utility for player i at terminal node x
- Then for any strategy profile c, the expected payoff for player i is:

$$u_i(c) = \sum_{x \in \Omega^*} P(x \mid c) \cdot w_i(x) \tag{4}$$

- This gives the **normal representation**:

$$\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$$

- Each player's payoff depends on the **entire strategy profile** c
- All probabilistic uncertainty is resolved by computing $P(x \mid c)$ over the tree

Convert Both Games to their Normal Forms

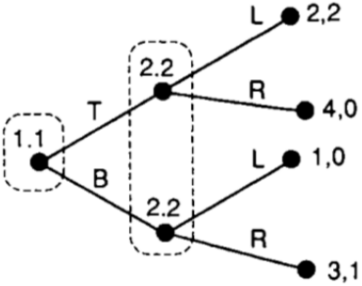


Figure 7: Game 3

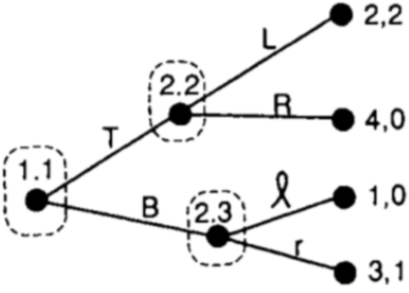


Figure 8: Game 4

Equivalence of Strategic-Form Games

- Utility numbers represent **preferences**, not cardinal values
- Two games are equivalent if they reflect **the same preferences**
- Utility functions $u(\cdot)$ and $\hat{u}(\cdot)$ are **equivalent** iff:

$$\hat{u}(c) = Au(c) + B, \quad A > 0$$

- Two strategic-form games Γ and $\hat{\Gamma}$ are **fully equivalent** iff:

$$\hat{u}_i(c) = A_i u_i(c) + B_i, \quad \forall c \in C, \forall i \in N$$

Implication of Full Equivalence

- Solution concepts must give the **same prediction** for equivalent games
- Each player i prefers strategy distribution μ over λ if:

$$\sum_{c \in C} \mu(c) u_i(c) \geq \sum_{c \in C} \lambda(c) u_i(c)$$

- In the transformed game $\hat{\Gamma}$, same preference holds if:

$$\sum_{c \in C} \mu(c) \hat{u}_i(c) \geq \sum_{c \in C} \lambda(c) \hat{u}_i(c)$$

Best-Response Equivalence (1/3)

- A weaker notion: players may have **same best responses** even if utilities differ
- Let C_{-i} be strategy combinations of all players **except** i:

$$C_{-i} = \prod_{j \in N \setminus \{i\}} C_j$$

- Given belief $\eta \in \Delta(C_{-i})$, player i chooses:

$$\arg\max_{d_i \in C_i} \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) u_i(e_{-i}, d_i)$$

Probability Distributions over Strategies (2/3)

- The set $\Delta(C_{-i})$ denotes all probability distributions over C_{-i}:

$$\Delta(C_{-i}) = \left\{ \eta : C_{-i} \to [0,1] \,\middle|\, \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) = 1 \right\}$$

- Each $\eta \in \Delta(C_{-i})$ assigns a probability to every possible combination of strategies chosen by players other than i
- The probabilities must be nonneg and sum to 1

Best-Response Equivalence (3/3)

- In $\hat{\Gamma}$, same rule applies with $\hat{u}_i$:

$$\arg\max_{d_i \in C_i} \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) \hat{u}_i(e_{-i}, d_i)$$

- Γ and $\hat{\Gamma}$ are **best-response equivalent** iff:

$$\arg\max_{d_i \in C_i} \sum_{e_{-i}} \eta(e_{-i}) u_i(e_{-i}, d_i) = \arg\max_{d_i \in C_i} \sum_{e_{-i}} \eta(e_{-i}) \hat{u}_i(e_{-i}, d_i)$$

for all $i \in N$ and all $\eta \in \Delta(C_{-i})$

Reduced Normal Representations

- Some strategic-form games can be simplified by removing **redundant strategies**
- Example: In an extensive-form game, if a node is never reached, actions taken at that node don't affect payoffs
- Such strategies can be **grouped or eliminated**

Payoff Equivalence

- Given game $\Gamma = (N, (C_i)_{i\in N}, (u_i)_{i\in N})$:
- Two strategies $d_i, e_i \in C_i$ are **payoff equivalent** iff:

$$u_j(c_{-i}, d_i) = u_j(c_{-i}, e_i), \quad \forall c_{-i} \in C_{-i},\ \forall j \in N$$

- That is, no player is ever affected by whether i plays d_i or e_i
- A **purely reduced normal representation** merges all payoff-equivalent strategies into one

Random Redundancy

- A **randomized strategy** $\sigma_i \in \Delta(C_i)$ is a probability distribution over C_i
- A **pure strategy** $c_i \in C_i$ is **randomly redundant** if:
  - There exists $\sigma_i \in \Delta(C_i)$ such that $\sigma_i(c_i) = 0$ and:

$$u_j(c_{-i}, c_i) = \sum_{e_i \in C_i} \sigma_i(e_i) u_j(c_{-i}, e_i), \quad \forall c_{-i} \in C_{-i}, \ \forall j \in N$$

- That is, c_i can be replaced by a **randomized mixture** without affecting any player's payoff

- If $\sigma_i(c_i) > 0$ were allowed, the condition would be trivial:
  - The degenerate mixture $\sigma_i(c_i) = 1$ always replicates c_i perfectly
- The requirement $\sigma_i(c_i) = 0$ forces replication by **genuinely other strategies**

Example (1/3)

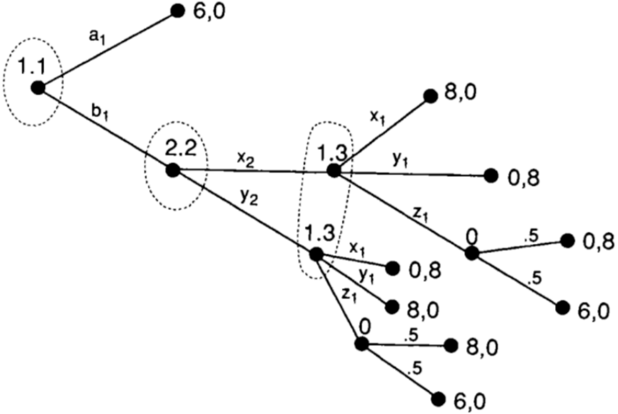


Figure 9: Game 5

Example (2/3)

C_1	C_2: x_2	C_2: y_2
a_1x_1	6,0	6,0
a_1y_1	6,0	6,0
a_1z_1	6,0	6,0
b_1x_1	8,0	0,8
b_1y_1	0,8	8,0
b_1z_1	3,4	7,0

Figure 10: Game 5

Example (3/3)

C_1	C_2: x_2	C_2: y_2
$a_1\cdot$	6,0	6,0
b_1x_1	8,0	0,8
b_1y_1	0,8	8,0
b_1z_1	3,4	7,0

Figure 11: Game 5

Strong Domination

- In a strategic-form game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$:
  - A strategy $d_i \in C_i$ is **strongly dominated** for player i if:

$$u_i(c_{-i}, d_i) < \sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i), \quad \forall c_{-i} \in C_{-i}$$

  for **some** randomized strategy $\sigma_i \in \Delta(C_i)$

- **A strongly dominated strategy can never be a best response under any belief**

Iterated Elimination of Strongly Dominated Strategies

- Let $C_i^{(1)}$ be the subset of strategies in C_i not strongly dominated
- Let $\Gamma^{(1)}$ be the game with $(C_i^{(1)})_{i \in N}$
- Define inductively:

$$\Gamma^{(k)} = \left(N, (C_i^{(k)})_{i \in N}, (u_i)_{i \in N}\right)$$

where $C_i^{(k)}$ is the set of strategies in $C_i^{(k-1)}$ that are not strongly dominated

- There exists a smallest K such that:

$$C_i^{(K)} = C_i^{(K+1)} = \dots \quad \text{for all } i \in N$$

Iteratively Undominated Strategies

- Let $C_i^{(\infty)} = C_i^{(K)}$ and $\Gamma^{(\infty)} = \Gamma^{(K)}$
- Then $C_i^{(\infty)}$ contains all strategies that are **iteratively undominated** in the strong sense
- Every strategy in $C_i^{(\infty)}$ is a best response to some belief over $C_{-i}^{(\infty)}$
- The set $(C_i^{(\infty)})_{i \in N}$ is maximal under this condition

Weak Domination

- A strategy $d_i \in C_i$ is **weakly dominated** for player i if:

$$u_i(c_{-i}, d_i) \leq \sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) \quad \forall c_{-i} \in C_{-i}$$

  and the inequality is strict for at least one $c_{-i} \in C_{-i}$
- Weak domination allows for a strategy to be a best response for some beliefs
- Iterative elimination of weakly dominated strategies is **order dependent** and may alter outcomes

Note

Strong domination: the strategy is *never* a best response. Weak domination: it is sometimes a best response but never strictly better than the dominating strategy. The distinction matters for solution concepts.

Multiagent Representation

Why Do We Need a Multiagent Representation?

- The strategic form bundles a player's entire plan into a single strategy
- This works well at the planning stage – but it creates a problem for **sequential rationality**

- Suppose a player reaches an information state that was supposed to have zero probability under the equilibrium
- Her strategy was formed before the game started, but now she is *actually at this node*
- Should she follow her original plan, or reconsider?

- The strategic form cannot answer this – it has no way to separate decisions at different nodes
- The **multiagent representation** separates decisions by information state, making sequential rationality checkable at each node independently

Multiagent Representation – Key Idea

- Instead of treating each player as a single decision-maker, we treat each **information state** as a separate agent
- These agents are called **temporary agents** – they share the same utility function as the original player, but act independently at their own node
- This matters because:
  - It allows us to ask whether behavior is optimal **at each node separately**
  - It is the natural foundation for **equilibrium refinements** such as sequential equilibrium and trembling-hand perfection
  - It makes off-path behavior explicit – the trembles of a player at one node are **independent** of trembles at another
- Think of each information state as a different "self" of the player, each making an independent decision but all sharing the same payoff function.

Multiagent Representation – Definition

- The **multiagent representation** maps an extensive-form game into a strategic-form game
- Each **information state** of a player is treated as a separate player (agent)
- These agents, called **temporary agents**, share the same preferences as the original player

Example (1/3)

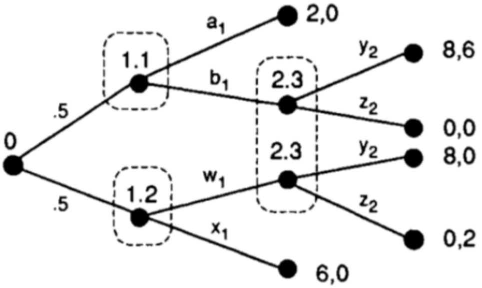


Figure 12: Game 6

Example (2/3)

- Normal Form

	C_2	
C_1	y_2	z_2
a_1w_1	5,0	1,1
a_1x_1	4,0	4,0
b_1w_1	8,3	0,1
b_1x_1	7,3	3,0

Example (3/3)

- Multiagent Representation

	y_2		z_2	
	w_1	x_1	w_1	x_1
a_1	5,5,0	4,4,0	1,1,1	4,4,0
b_1	8,8,3	7,7,3	0,0,1	3,3,0

What the Multiagent Representation Reveals

- In the normal form, Player 1's strategies bundle choices at all her information states together
- In the multiagent representation, each information state is a **separate player**
- This separation reveals something the normal form hides:

- A Nash equilibrium of the multiagent game requires each agent to be **independently optimal** at her own node
- This is exactly the condition of **sequential rationality**
- An equilibrium of the normal form need not satisfy this – a player may be best-responding overall while acting suboptimally at a node that is never reached

- The multiagent representation is the bridge between the normal form and the extensive-form refinements we will study later (sequential equilibrium, perfect equilibrium). It makes "off-path rationality" a first-class concept.

Common Knowledge and Bayesian Games

Common Knowledge

- A fact is **common knowledge** among players if:
  - Every player knows it,
  - Every player knows that every player knows it,
  - Every player knows that every player knows that every player knows it,
  - and so on...
- Formally, every statement of the form

$$(\text{every player knows that})^k \text{ every player knows it}$$

  is true for all $k = 0, 1, 2, \dots$
- **Private information**: information held by a player that is *not* common knowledge among all players

Common Knowledge – An Illustration

- Suppose two people are in a room and both **see a fire alarm go off**
- Each knows there is a fire; each knows the other knows; and so on – this is common knowledge

- Now suppose each receives a **private text message** saying "there is a fire"
- Each knows there is a fire, but neither knows whether the other received the message
- This is *not* common knowledge – the public alarm creates it, the private message does not

> **Note**
>
> This distinction matters in games: players can coordinate on an equilibrium only when the equilibrium itself is common knowledge. Private signals, even about the same event, may not support coordination.

Intelligence Assumption

- In game theory, we assume players are **intelligent**:
  - Whatever we (the analysts) know about the **structure** of the game, the players also know
  - Players know the model, the strategy sets, and the payoff functions
  - Furthermore, they know that they all know the model, and so on

> **Note**
>
> The intelligence assumption means the **game structure** is common knowledge – not the realization of private information. Player 2 knows Player 1 drew a card; she does not know which color was drawn.

- Therefore, when constructing a model of a game, the **game structure itself must be common knowledge**

Bayesian Games – Incomplete Information

- A game has **incomplete information** when players know different things at the time they start planning their moves
- One or more players may have **private information** that others do not know

- Example: A card game becomes one of incomplete information if Player 1 looks at a card before learning how it will be used

- Such games are common in practice

Modeling Private Information

- Often, players have private information they have known for a long time
- It may feel unnatural to define the game as starting **before** they learned this
- Some private information is part of a player's identity (e.g., gender, language, risk preferences)
- This initial private information is called the player's **type**

Representing Bayesian Games

- We can model these games using a **historical chance node**:
    - This node determines the players' types before actions begin
- But this can feel artificial if types are basic to identity
- So, we admit that players **start** the game already knowing their types

Strategic Form vs. Bayesian Form

- In standard extensive-form games, players plan their moves at the **root**, before any private info is learned
- In Bayesian games, this assumption is **invalid**:
  - The root is before type information is known
  - But players begin planning **after** knowing their types
- Harsanyi (1967–68) introduced a solution

Harsanyi's Bayesian Form

- The **Bayesian form** generalizes the strategic form to handle private information
- Players do not choose strategies before learning their type
- This form is:
    - Simpler than the full extensive form
    - More realistic than assuming full information at the root
- Bayesian games are strategic-form games where each player has a **type**, and strategies are chosen **after** types are realized

Components of a Bayesian Game

- To define a Bayesian game, we specify:
  - A set of players: N
  - For each player $i \in N$:
    - A set of actions: C_i
    - A set of types: T_i
    - A probability function: p_i
    - A utility function: u_i
- We define:

$$C = \prod_{i \in N} C_i, \quad T = \prod_{i \in N} T_i$$

  - C: all possible action profiles
  - T: all possible type profiles

Beliefs About Other Players' Types

- Let $T_{-i} = \prod_{j \in N \setminus \{i\}} T_j$
- The probability function p_i maps each $t_i \in T_i$ to a belief over T_{-i}:

$$p_i(\cdot \mid t_i) \in \Delta(T_{-i})$$

- So $p_i(t_{-i} \mid t_i)$ is player i's belief that the others have types t_{-i}, given his own type t_i

Utility Function in a Bayesian Game

- For each player i, the utility function is:

$$u_i : C \times T \to \mathbb{R}$$

- It assigns a payoff to each combination of:
  - Action profile $c \in C$
  - Type profile $t \in T$
- This payoff is on a von Neumann–Morgenstern utility scale

Formal Definition

Definition: Bayesian Game

A Bayesian game is:

$$\Gamma^b = (N, (C_i)_{i \in N}, (T_i)_{i \in N}, (p_i)_{i \in N}, (u_i)_{i \in N})$$

The game is **finite** if all sets N, C_i, and T_i are finite. It is **common knowledge** that each player knows his own type and the full structure of the game.

Actions vs. Strategies

- In a Bayesian game, we speak of **actions**, not strategies
- An **action** is a move chosen after the player learns their type
- A **strategy** (in extensive form) covers all contingencies before knowing the type
- So, a strategy for player i is a function:

$$s_i : T_i \to C_i$$

assigning an action for each possible type

Example – A Simple Card Game as a Bayesian Game

- Suppose Player 1 knows the color of a card at the start of the game
- Then the game has **incomplete information** and can be written in Bayesian form

Formal Representation

- Set of players: $N = \{1, 2\}$
- Types:
  - $T_1 = \{1.\text{a}, 1.\text{b}\}$ (red or black card)
  - $T_2 = \{2\}$ (no private info)
- Actions:
  - $C_1 = \{R, F\}$ (raise or fold)
  - $C_2 = \{M, P\}$ (meet or pass)

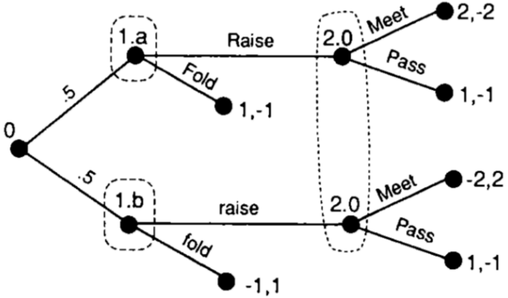


Figure 13: Game 2

Beliefs

- Player 2 believes the card is red or black with equal probability:

$$p_2(1.\mathsf{a} \mid 2) = 0.5 = p_2(1.\mathsf{b} \mid 2)$$

- Player 1 knows Player 2's type with certainty:

$$p_1(2 \mid 1.\mathsf{a}) = 1 = p_1(2 \mid 1.\mathsf{b})$$

Utility Functions

- Utility functions are of the form:

$$u_i(c, t), \quad \text{where } c = (c_1, c_2),\ t = (t_1, t_2)$$

- That is, each player's payoff depends on both the **action profile** and the **type profile**

- **Example entries** (card game):
  - $u_1((R, M), (1.\text{a}, 2)) = 2$ (Player 1 raises, Player 2 meets, card is red – Player 1 wins)
  - $u_1((R, M), (1.\text{b}, 2)) = -2$ (same actions, card is black – Player 2 wins)

Bayesian Representation

t_1 = 1.a

	M	P
R	2,−2	1,−1
F	1,−1	1,−1

t_1 = 1.b

	M	P
R	−2,2	1,−1
F	−1,1	−1,1

Figure 14: Bayesian Representation

Type-Agent Representation (Harsanyi–Selten)

- Harsanyi (1967–68), inspired by Selten, introduced a way to express any Bayesian game Γ^b in **strategic form**
- This method is called the **type-agent representation** (also known as the **Selten game** or **posterior-lottery model**)

Key Idea

- In this representation:
  - Each **type** of each player is treated as a separate agent
  - So, there is **one agent per type** in the original Bayesian game
- We assume type sets are disjoint:

$$T_i \cap T_j = \emptyset \quad \text{for } i \neq j$$

- Let the set of all agents be:

$$T^* = \bigcup_{i \in N} T_i$$

Strategies and Actions

- Each agent $t_i \in T_i$ chooses from the same actions as player i:

$$D_{t_i} = C_i$$

- Agent t_i is responsible for the action taken **if player** i **has type** t_i in the Bayesian game

Expected Utility

- Agent t_i's payoff is the **expected utility** of player i given t_i is their type
- Formally, for any profile of decisions $d = (d(s))_{s \in T^*}$:

$$v_{t_i}(d) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i) \cdot u_i\left((d(t_j))_{j \in N}, (t_j)_{j \in N}\right)$$

- The outcome depends on:
  - Actions selected by all agents
  - The type profile $(t_j)_{j \in N}$

Type-Agent Representation – Normal Form

- Agents: $T^* = \{1.a,\ 1.b,\ 2\}$
- Rows give the action of agent $1.a$
- Columns give the action pair $(d_{1.b}, d_2)$
- Entries are $(v_{1.a},\ v_{1.b},\ v_2)$
- Agent 2 averages over Player 1's type using $p = 0.5$

	$d_2 = M$		$d_2 = P$	
	$R_{1.b}$	$F_{1.b}$	$R_{1.b}$	$F_{1.b}$
$R_{1.a}$	$2,\ -2,\ 0$	$2,\ 1,\ -0.5$	$1,\ 1,\ -1$	$1,\ -1,\ 0$
$F_{1.a}$	$1,\ -2,\ 0.5$	$1,\ 1,\ 0$	$1,\ 1,\ -1$	$1,\ -1,\ 0$

Modeling Games with Incomplete Information

- Game theory gives us a general framework to model strategic situations
- But modeling **incomplete information** is difficult in practice
- Main issue: players' beliefs are often **subjective probabilities**
  - What one player believes about another's belief becomes complex

Example – Trivia Quiz Game

- We modify the card game into a **trivia quiz game**:
  1. Both players put $1 into the pot
  2. A question is randomly drawn and announced to both
  3. Player 1 can **Raise** or **Fold**
  4. If Player 1 raises, Player 2 can **Meet** (add $1) or **Pass**
  5. If Player 2 meets, Player 1 answers the question
- Player 1 wins if:
  - They fold and keep $1, or
  - They raise, Player 2 passes, or
  - They raise, Player 2 meets, and Player 1 answers correctly
- Otherwise, Player 2 wins

Belief Complexity

- After the question is announced, Player 2 is unsure whether Player 1 knows the answer
- Let Q be Player 2's **subjective probability** that Player 1 knows the answer:

$$Q \in [0, 1]$$

- But Player 1 may be unsure about Player 2's Q
- Then Player 1 needs a **distribution over** Q
- And Player 2 may be unsure about that – so a **distribution over distributions**!
- This leads to **higher-order beliefs** and infinite regress

Universal Belief Space

- Can any Bayesian game represent **all** possible beliefs?
- Mertens & Zamir (1985) proved: Yes, using a **universal belief space**
- It is a belief space large enough to encode **every level** of beliefs:
  - About the state
  - About others' beliefs
  - About others' beliefs about your beliefs, etc.

The Universal Belief Space – A Double-Edged Sword (1/2)

Result (Mertens–Zamir 1985)

A universal belief space T^* exists that encodes every possible hierarchy of beliefs – Harsanyi's type-space approach is without loss of generality.

- This is reassuring: **any** belief configuration can be represented
- But the richness of T^* creates a fragility problem:

- Rationalizability is the robust baseline
  - It does not rule out much; it only eliminates actions inconsistent with rationality and beliefs in rationality
- Stronger concepts, such as Bayesian Nash equilibrium, make sharper predictions
- Weinstein–Yildiz (2007) show that these extra predictions may be fragile
- Any action still allowed by rationalizability can become **uniquely predicted** after a small perturbation of higher-order beliefs

The Universal Belief Space – A Double-Edged Sword (2/2)

Concept	Robust in T^*?	Why?
Iterated strict dominance	**Yes**	Stable under belief perturbations
Iterated weak dominance	**No**	Sensitive to zero-probability beliefs
Nash equilibrium	**Partial**	Exists, but selection is fragile
Nash refinements	**No**	Sensitive to higher-order beliefs

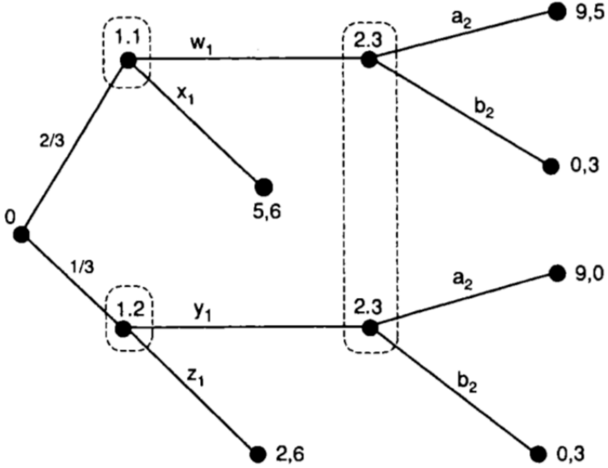
Limitations and Practical Modeling

- The universal belief space is mathematically valid but **huge** and **intractable**
- In practice, we need to work with **simplified belief structures**
- We rely on **common knowledge** to:
  - Keep type sets manageable
  - Focus on core strategic uncertainty
- So while theory guarantees generality, practice requires simplification

Required Reading

- **Myerson (1997)** Chapter 2 – Basic Models
- *Optional:*
  - Mertens, J.F., and Zamir, S. (1985). Formulation of Bayesian Analysis for Games with Incomplete Information. *International Journal of Game Theory*, 14(1), 1–29.
  - Weinstein, J., and Yildiz, M. (2007). A Structure Theorem for Rationalizability with Application to Robust Predictions of Refinements. *Econometrica*, 75(2), 365–400.

Example



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