Microeconomics IV (Game Theory)

Lecture 5 - Some Refinements of Equilibrium in Strategic Form

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2026

Refinements of Nash Equilibrium

Introduction

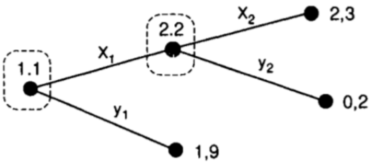
- Any extensive-form game can be represented in strategic form.
- There are multiple representations:
    - **Normal representation**
    - **Reduced normal representation**
    - **Multiagent representation**
- These forms raised the hope that all principles of game-theoretic analysis could be applied in the simpler strategic form.

Strategic vs Extensive Form

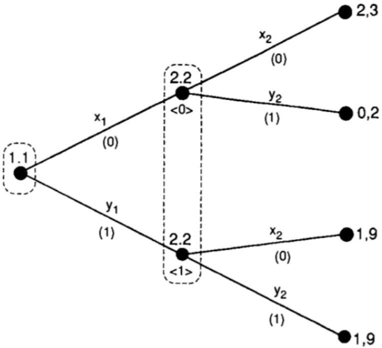
- However, this hope does **not always hold**.
- It is easy to construct extensive-form games that:
  - Are identical in strategic form (under any representation),
  - But yield **different sets of sequential equilibria**.

Example 1: Same Strategic Form, Different Equilibria

Game 1



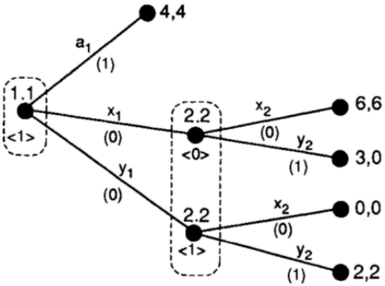
Game 2



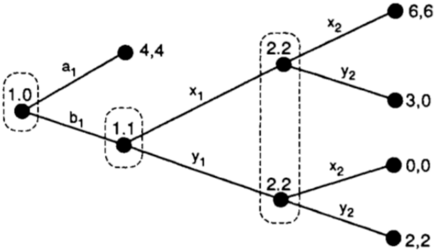
- Both games have **identical strategic representations**.
- But $([y_1],[y_2])$ is a **sequential equilibrium** in Game 2, not in Game 1.
- Extensive form matters for belief formation and rationality.

Example 2: Games 3 and 4

Game 3



Game 4



- **Game 3** and **Game 4** have the *same reduced normal form* (up to relabeling).
- But: Game 3 admits a sequential equilibrium with Player 1 choosing a_1.
- In Game 4, no such sequential or subgame-perfect equilibrium exists.

Why the Difference?

- In **Game 4**, Player 2 observes that Player 1 did *not* choose a_1.
- So Player 2 must infer Player 1 chose between x_1 and y_1 *intentionally*.
- Rationality $\Rightarrow$ Player 1 would choose the better move between x_1 and y_1 $\Rightarrow$ surely x_1.
- So Player 2 chooses x_2.
- This logic prevents a_1 from being part of any equilibrium.

Why Refinements Are Needed

- We saw that:
  - Strategic-form equivalence can mask **differences in sequential behavior**.
  - Some sequential equilibria include **weakly dominated strategies**.
  - Some games have **no reasonable sequential equilibrium**.
- To resolve these issues, we need **refinements of Nash equilibrium** that:
  - Apply in strategic form.
  - Reflect **sequential rationality** from the extensive form.

Desirable Properties of a Refined Equilibrium

- A refined equilibrium should:
  - Always **exist** for any finite game.
  - Exclude outcomes involving **dominated strategies**.
  - Eliminate equilibria that **cannot be justified by any belief system**.
  - Preserve equivalence across:
    - Normal form,
    - Reduced normal form,
    - Multiagent representations.

Core Idea

Note

A good refinement of Nash equilibrium should give the same solution across all strategic-form representations of the same extensive-form game.

- This ensures that solution concepts are **representation-invariant**.

Perfect Equilibrium

Perfect Equilibrium

- One such refinement: **Perfect Equilibrium** (Selten, 1975).
- Key features:
  - Based on **trembles** (small mistakes with positive probability).
  - Eliminates equilibria that rely on **non-credible threats**.
  - Preserves rationality across strategic and extensive forms.
- Especially robust when applied through the **multiagent representation**.

Perfect Equilibrium

Perfect Equilibrium

Let $\Gamma = (N, (C_i)_{i\in N}, (u_i)_{i\in N})$ be a finite game in strategic form. A mixed-strategy profile $\sigma \in \times_{i\in N}\Delta(C_i)$ is a **perfect equilibrium** if there exists a sequence $(\hat{\sigma}^k)_{k=1}^{\infty}$ such that:

- $\hat{\sigma}^k \in \times_{i\in N}\Delta^0(C_i), \quad \forall k \in \{1, 2, 3, \dots\} \qquad (1)$
- $\lim_{k\to\infty} \hat{\sigma}_i^k(c_i) = \sigma_i(c_i), \quad \forall i \in N,\ \forall c_i \in C_i \qquad (2)$
- $\sigma_i \in \arg\max_{\tau_i \in \Delta(C_i)} u_i(\hat{\sigma}_{-i}^k, \tau_i), \quad \forall i \in N \qquad (3)$

- For any finite set Z, $\Delta^0(Z) = \{q \in \Delta(Z) \mid q(z) > 0,\ \forall z \in Z\}$ – the set of probability distributions assigning strictly positive probability to every element in Z.

Interpretation

- (1): Each $\hat{\sigma}^k$ assigns **positive probability to every pure strategy**.
- (2): The sequence converges to the equilibrium σ.
- (3): Each σ_i is a **best response** to $\hat{\sigma}^k_{-i}$.

> **Note**
>
> Perfect Equilibrium refines Nash by ensuring robustness to small trembles in opponents' strategies.

Perfect Equilibrium Implies Nash

- Every perfect equilibrium is also a **Nash equilibrium**.
- Why? Because:
  - The **best-response correspondence** is upper hemicontinuous.
  - Limits of perturbed equilibria still satisfy the best-response condition.

- Formally:

$$\sigma_i \in \arg\max_{\tau_i \in \Delta(C_i)} u_i(\sigma_{-i}, \tau_i), \quad \forall i \in N$$

Note

In a perfect equilibrium, strategies remain best responses even when every pure strategy is played with positive probability in the perturbation sequence.

Connection to Sequential Equilibrium

- Both **perfect** and **sequential** equilibrium test σ against **perturbed strategies**.
- In sequential equilibrium:
  - Perturbations help generate beliefs at zero-probability information sets.
  - Beliefs are used in the optimality condition.
- In perfect equilibrium:
  - Perturbations are directly used in the **payoff comparison** (condition 3).
  - No need to construct beliefs – just evaluate payoffs from trembles.

Note

In the multiagent representation, condition (3) is stronger than the sequential rationality condition. Therefore, $\Rightarrow$ Every perfect equilibrium of the multiagent game Γ^e is a sequential equilibrium of Γ^e.

Recall Notation – Game, Players, Nodes

Symbol	Meaning
Γ^e	Extensive-form game with **perfect recall**
$i,\ s \in S_i$	Player i and one of his information states s
D_s	Set of moves available at state s
Y_s	Set of **decision nodes** in information state s
$X(s)$	Set of terminal nodes that do **not** follow any node in Y_s – the **bypass set** (plays that never reach s)
$w_i(x)$	Payoff to player i at terminal node x

- $X(s)$ is the one piece worth pausing on: it collects exactly the plays where i's move at s is **irrelevant**, because the game never visits s.

Recall Notation – Strategies, Beliefs, Payoffs

Symbol	Meaning
$\hat{\sigma}^k$	A sequence of **fully mixed** (strictly positive) profiles supporting σ as a perfect equilibrium of the multiagent rep.
$\rho_s \in \Delta(D_s)$	An alternative local move distribution at state s
$\bar{P}(y \mid \sigma)$	Probability that node y is reached under profile σ
$U_i(\sigma \mid y)$	Expected **continuation** payoff to i from node y onward
v_s	Payoff of player i's **agent at state** s in the multiagent rep – an **ex ante** (whole-tree) expectation, *not* a conditional one
$\hat{\pi}_s^k,\ \pi_s$	Beliefs from $\hat{\sigma}^k$ by Bayes; $\pi_s = \lim_k \hat{\pi}_s^k$

Theorem: Perfect $\Rightarrow$ Sequential

Theorem 1

Suppose Γ^e is an extensive-form game with **perfect recall** and σ is a **perfect equilibrium** of the **multiagent representation** of Γ^e. Then there exists a belief vector π such that (σ, π) is a **sequential equilibrium** of Γ^e.

- Links perfect equilibrium (a strategic-form refinement) to sequential equilibrium (an extensive-form refinement).
- Both are built from the **same trembling-hand limit** – perfection's node-by-node best response is exactly the raw material sequential rationality needs.

Proof Sketch (1/3) – Beliefs from Trembles

Note

Goal: From a perfect equilibrium σ of the multiagent rep, build beliefs π so that (σ, π) is a sequential equilibrium.

- Perfection gives a sequence of **fully mixed** profiles $\hat{\sigma}^k \to \sigma$.
- For each k and each node $y \in Y_s$, define beliefs by Bayes' rule:

$$\hat{\pi}_s^k(y) = \frac{\bar{P}(y \mid \hat{\sigma}^k)}{\sum_{z \in Y_s} \bar{P}(z \mid \hat{\sigma}^k)}$$

- Since $\hat{\sigma}^k$ is strictly positive, $\bar{P}(y \mid \hat{\sigma}^k) > 0$ for every y, so the denominator is **positive** and the ratio is well-defined.
- Set $\pi_s(y) = \lim_{k \to \infty} \hat{\pi}_s^k(y)$. By construction π is **fully consistent** with σ.

Proof Sketch (2/3) – Splitting the Agent's Payoff

- The agent-s payoff against $\hat{\sigma}^k$, when playing ρ_s, splits into plays **through** s and plays **bypassing** s:

$$v_s(\hat{\sigma}^k_{-i,s}, \rho_s) = \sum_{y \in Y_s} \bar{P}(y \mid \hat{\sigma}^k)\, U_i(\hat{\sigma}^k_{-i,s}, \rho_s \mid y) \; + \sum_{x \in X(s)} \bar{P}(x \mid \hat{\sigma}^k)\, w_i(x)$$

- **Key fact:** $\bar{P}(y \mid \hat{\sigma}^k_{-i,s}, \rho_s) = \bar{P}(y \mid \hat{\sigma}^k)$ – reaching $y \in Y_s$ depends only on agents who move **before** s, not on ρ_s.
- The bypass term $\sum_{x \in X(s)} \bar{P}(x \mid \hat{\sigma}^k)\, w_i(x)$ is **constant in** ρ_s – those plays never reach s.

Proof Sketch (2/3, cont.) – Factoring out Beliefs

- Multiply and divide the first sum by the reach-probability of s:

$$v_s = \underbrace{\left(\sum_{y \in Y_s} \hat{\pi}_s^k(y)\, U_i(\hat{\sigma}_{-i,s}^k, \rho_s \mid y) \right)}_{\text{belief-weighted continuation}} \underbrace{\left(\sum_{z \in Y_s} \bar{P}(z \mid \hat{\sigma}^k) \right)}_{\text{reach prob. of } s > 0} + \underbrace{\sum_{x \in X(s)} \bar{P}(x \mid \hat{\sigma}^k)\, w_i(x)}_{\text{const. in } \rho_s}$$

- So v_s is a **strictly increasing affine transformation** of the conditional, belief-weighted objective – with coefficients **independent of** ρ_s.
- $\to$ The two objectives have the **same maximizers**.

Proof Sketch (3/3) – Sequential Rationality

- Perfection means σ_s best-responds in the agent's payoff at every k:

$$\sigma_s \in \arg\max_{\rho_s \in \Delta(D_s)} v_s(\hat{\sigma}^k_{-i,s}, \rho_s)$$

- By the affine equivalence, equivalently:

$$\sigma_s \in \arg\max_{\rho_s \in \Delta(D_s)} \sum_{y \in Y_s} \hat{\pi}^k_s(y)\, U_i(\hat{\sigma}^k_{-i,s}, \rho_s \mid y)$$

- Let $k \to \infty$. By **upper-hemicontinuity** of the best-response correspondence (limits of maximizers maximize the limit):

$$\sigma_s \in \arg\max_{\rho_s \in \Delta(D_s)} \sum_{y \in Y_s} \pi_s(y)\, U_i(\sigma_{-i,s}, \rho_s \mid y)$$

- This is exactly the sequential-rationality condition. With π fully consistent (Step 1), (σ, π) is a sequential equilibrium. ■

Selten's Definition (1975)

Note

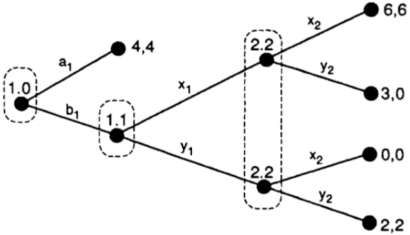
Selten (1975) defined a **perfect equilibrium** of an extensive-form game to be any perfect equilibrium of its **multiagent representation**.

- This **contrasts** with the definition of **Nash** equilibrium:
  - A Nash equilibrium of an extensive-form game is defined as a Nash equilibrium of its multiagent representation,
  - **Only if** the corresponding mixed strategy profile is also a Nash equilibrium in the **normal representation**.
- Perfect equilibrium relies only on the multiagent structure, not the normal form.

Example 3

- Consider **Game 4** and its norml representation:

Game 4



Normal Representation

C_1	C_2: x_2	y_2
a_1x_1	4,4	4,4
a_1y_1	4,4	4,4
b_1x_1	6,6	3,0
b_1y_1	0,0	2,2

Perfect Equilibrium in the Normal Form

- The strategy profile $([a_1x_1],[y_2])$ is a **perfect equilibrium** in the normal representation.
- To see this, construct the perturbed strategy sequence:

$$\hat{\sigma}^k = ((1-\varepsilon)[a_1x_1] + .1\varepsilon[a_1y_1] + .1\varepsilon[b_1x_1] + .8\varepsilon[b_1y_1], \quad \varepsilon[x_2] + (1-\varepsilon)[y_2])$$

with $\varepsilon = 1/(k+2)$.

Numerical Justification

- When $\varepsilon \leq \frac{1}{3}$:
    - Player 1:

      $$\text{Payoff from } a_1x_1 = 4 \geq 6\varepsilon + 3(1-\varepsilon)$$

      So $[a_1x_1]$ is a best response to $\varepsilon[x_2] + (1-\varepsilon)[y_2]$.
    - Player 2:
        - If b_1y_1 has higher probability (more than 3 times larger) than b_1x_1, then y_2 is optimal.

Note

Hence, $(\hat{\sigma}^k)$ satisfies (1)–(3), so $([a_1x_1], [y_2])$ is a perfect equilibrium in the normal form.

Multiagent Representation Breakdown

- In the multiagent form, equilibrium must hold **at each node**, for each **agent**.
- Suppose Player 1's agent at node 1.0 plays b_1 with positive probability.
- Then at node 1.1, the unique best response must be x_1:

$$\lim_{k \to \infty} \hat{\sigma}^k_{1.1}(x_1) = 1, \quad \hat{\sigma}^k_{1.1}(y_1) = 0$$

- This implies:
  - $b_1 x_1$ occurs with high probability,
  - So Player 2 responds with x_2, not y_2.

Consequences for Perfect Equilibrium

- If x_1 and x_2 are highly likely:
  - The best response at 1.0 is b_1 (since $6 > 4$),
  - So Player 1 plays b_1 with high probability.
- Therefore, any perfect equilibrium in the **multiagent representation** must satisfy:

$$([b_1], [x_1], [x_2])$$

Note

⇒ The strategy profile $([a_1 x_1], [y_2])$ is a perfect equilibrium in the normal form, but **not** in the multiagent (or extensive-form) representation.

Existence of Perfect and Sequential Equilibria

Theorem 2

For any finite game in strategic form, there exists at least one perfect equilibrium.

- Theorems **1** and **2** immediately imply the existence theorem for sequential equilibria.

Example 4 - Perfect Equilibrium in Normal Games

- Consider a 3-player game with:
  - Player 1 strategy set: $S_1 = \{U, D\}$
  - Player 2 strategy set: $S_2 = \{L, R\}$
  - Player 3 strategy set: $S_3 = \{X, Y\}$

Player 3 plays X

	L	R
U	$(1, 1, 1)$	$(1, 0, 1)$
D	$(1, 1, 1)$	$(0, 0, 1)$

Player 3 plays Y

	L	R
U	$(1, 1, 0)$	$(0, 0, 0)$
D	$(0, 1, 0)$	$(1, 0, 0)$

Example 4 - Perfect Equilibrium in Normal Games

- Perform iterated deletion of weakly dominated strategies. What are the strategies that survive this process?
- **Player 2**:
  - Strategy L is strictly dominant over R
  - Keep L, delete R
- **Player 3**:
  - Strategy X is strictly dominant over Y
  - Keep X, delete Y
- **Player 1**:
  - No strict or weak domination between U and D
  - No strategies deleted for player 1
- **Surviving Strategies:** $\{\{U, D\}, \{L\}, \{X\}\}$

Example 4 - Perfect Equilibrium in Normal Games

- Find all Nash equilibria (both pure and mixed). Does any of the Nash equilibrium involve weakly dominated strategies?
- **Player 2** plays strictly dominant strategy L
- **Player 3** plays strictly dominant strategy X
- Given (L, X), Player 1 is **indifferent** between U and D
- So any mixed strategy of Player 1 between U and D is a best response
- **Nash Equilibria:**

$$(pU + (1-p)D,\ L,\ X), \quad \text{for any } p \in [0,1]$$

- **None of these NE involve weakly dominated strategies**

Example 4 - Perfect Equilibrium in Normal Games

- Focus on the pure strategy Nash equilibria. Are they trembling hand perfect?
- The game has two pure strategy Nash equilibria:

$$(U, L, X) \quad \text{and} \quad (D, L, X)$$

- We will analyze whether these are **trembling-hand perfect** (THP) equilibria (alternative name).

Example 4 - Perfect Equilibrium in Normal Games

- Let the perturbed (completely mixed) strategy profile be:

$$\sigma_1(\varepsilon_1) = \varepsilon_1 U + (1-\varepsilon_1)D$$
$$\sigma_2(\varepsilon_2) = (1-\varepsilon_2)L + \varepsilon_2 R$$
$$\sigma_3(\varepsilon_3) = (1-\varepsilon_3)X + \varepsilon_3 Y$$

- As $\varepsilon \to (0,0,0)$, the profile $\sigma(\varepsilon) \to (D, L, X)$

Example 4 - Perfect Equilibrium in Normal Games

- Checking if (D, L, X) is THP
- Compute expected payoffs for Player 1:

$$u_1(U, \sigma_2, \sigma_3) = 1 - \varepsilon_2\varepsilon_3 \quad (1)$$

$$u_1(D, \sigma_2, \sigma_3) = 1 - \varepsilon_2 - \varepsilon_3 + 2\varepsilon_2\varepsilon_3 \quad (2)$$

- For D to be a best response:

$$1 - \varepsilon_2 - \varepsilon_3 + 2\varepsilon_2\varepsilon_3 \geq 1 - \varepsilon_2\varepsilon_3 \Rightarrow 3\varepsilon_2\varepsilon_3 \geq \varepsilon_2 + \varepsilon_3$$

- But as $\varepsilon_i \to 0$ for $i = 2, 3$, this inequality is eventually violated.
- For example, if $\varepsilon_2 \in \left(0, \frac{1}{3}\right)$, the inequality fails regardless of ε_3.
- $\Rightarrow D$ cannot be a best response to the sequence of totally mixed strategies of players 2 and 3.

Example 4 - Perfect Equilibrium in Normal Games

- Is (U, L, X) THP?
- From equations (1) and (2), we now check if U can be a best response.
- We want a sequence of $(\varepsilon_2, \varepsilon_3) \to 0$ such that U is better than D.

- Condition for U being a best response:

$$3\varepsilon_2\varepsilon_3 \leq \varepsilon_2 + \varepsilon_3 \quad \text{or} \quad 3\varepsilon_2 \leq 1 + \frac{\varepsilon_2}{\varepsilon_3}$$

- It suffices to let ε_2 be any sequence:
  - bounded above by $\frac{1}{3}$, and
  - converging to 0
- Let ε_3 be any sequence converging to 0
- Hence, U can be a best response in the limit, so (U, L, X) is **trembling-hand perfect**.

Implication: THP in Multi-Player Games

- In 2-player games:
  THP $\Leftrightarrow$ no weakly dominated strategies in NE
- In 3+ player games:
  THP can rule out NE that use only undominated strategies

Note

This game shows that some pure-strategy Nash equilibria – like (D, L, X) – are **not trembling-hand perfect**, even though they involve no weakly dominated strategies.

Why Trembling-Hand Perfection Makes Sense

- The two equilibria (U, L, X) and (D, L, X) give **identical on-path outcomes** – Player 1 is indifferent.
- But indifference holds only on the knife-edge where Players 2 and 3 play L, X with probability **exactly** 1.

- **Once we admit that hands shake**, the tie breaks:
  - U wins in the **single-tremble** cells (R, X) and (L, Y) – first order, $\sim \varepsilon_2 + \varepsilon_3$.
  - D wins only in the **double-tremble** cell (R, Y) – second order, $\sim \varepsilon_2\varepsilon_3$.
- Single mistakes are far likelier than simultaneous ones, so U is the **safe** response.

- *THP is just this intuition made precise:* don't merely best-respond when everyone plays perfectly – best-respond when everyone *almost* does. The catch: it bounds only the **size** of mistakes, not their **relative** likelihood. Constraining *that* – worse mistakes must be rarer – is the step from perfect to **proper** equilibrium.

Proper Equilibrium

Proper Equilibria

- Perfect equilibrium is defined using the **multiagent representation** of extensive-form games.
- However, even games with the **same reduced normal form** (e.g., Games 3 and 4) may be distinguished by perfect equilibrium.
- For example:
  - Both $([a_1], [y_2])$ and $([x_1], [x_2])$ are perfect equilibria in the multiagent version of **Game 3**
  - But only $([b_1], [x_1], [x_2])$ is perfect equilibrium in the multiagent form of **Game 4**

Motivation for Proper Equilibria

- This sensitivity motivates a refinement that **respects sequential rationality**, even when applied to the normal form.
- We want a solution concept that:
    - Selects equilibria that **can be extended to sequential equilibria**
    - Does **not distinguish games** based on artifacts of representation

Note

Proper equilibria form a **subset of perfect equilibria**, and always correspond to equilibria that can be extended to **sequential equilibria**.

ε-Perfect Equilibrium

- Let $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$ be a finite normal-form game.
- Let $\Delta^0(C_i)$ be the set of completely mixed strategies for player i.
- A strategy profile $\sigma \in \times_{i \in N} \Delta^0(C_i)$ is an ε-perfect equilibrium if:

$$\text{For all } i \in N \text{ and all } c_i \in C_i: \quad \text{if } c_i \notin \arg\max_{e_i \in C_i} u_i(\sigma_{-i}, [e_i]), \text{ then } \sigma_i(c_i) < \varepsilon$$

- In an ε-**perfect equilibrium**, players are allowed to make small mistakes.

Limit Characterization of Perfect Equilibrium

- A strategy profile $\bar{\sigma}$ is a perfect equilibrium if and only if there exists a sequence $(\varepsilon(k), \hat{\sigma}^k)_{k=1}^{\infty}$ such that:

$$\lim_{k\to\infty} \varepsilon(k) = 0, \quad \lim_{k\to\infty} \hat{\sigma}_i^k(c_i) = \bar{\sigma}_i(c_i) \quad \text{for all } i \in N, \text{ and all } c_i \in C_i$$

and each $\hat{\sigma}^k$ is an $\varepsilon(k)$-perfect equilibrium

Note

Perfect equilibrium is the limit of a sequence of ε-perfect equilibria as $\varepsilon \to 0$, where players assign vanishing probability to suboptimal pure strategies.

ε-Proper Equilibrium

- Let $\sigma \in \times_{i \in N} \Delta^0(C_i)$ be a completely mixed strategy profile.
- σ is an ε-proper equilibrium if:

  For all $i \in N$ and all $c_i, e_i \in C_i$: if $u_i(\sigma_{-i}, [c_i]) < u_i(\sigma_{-i}, [e_i])$, then $\sigma_i(c_i) \leq \varepsilon \cdot \sigma_i(e_i)$

- That is, **worse strategies must be played with much smaller probability** than better ones.

Proper Equilibrium (Limit Characterization)

- A strategy profile $\bar{\sigma}$ is a **proper equilibrium** if there exists a sequence $(\varepsilon(k), \sigma^k)$ such that:

$$\lim_{k\to\infty} \varepsilon(k) = 0, \quad \lim_{k\to\infty} \sigma_i^k(c_i) = \bar{\sigma}_i(c_i), \quad \forall i \in N,\ \forall c_i \in C_i$$

- And each σ^k is an $\varepsilon(k)$-proper equilibrium.

Intuition

- Every pure strategy gets positive probability.
- Strategies that are **clear mistakes** (i.e., strictly worse than others) receive **vanishingly small probability**.
- A strictly worse strategy must be played with an asymptotically smaller probability than a strictly better strategy.
- Proper equilibrium is a refinement of perfect equilibrium:
  - **Every proper equilibrium is perfect.**
  - **Not every perfect equilibrium is proper.**
- *Proper equilibrium ensures robustness to mistakes: the worse the mistake, the smaller the chance a rational player makes it.*

Existence of Proper Equilibrium

Theorem 3

For any finite game in strategic form, the set of proper equilibria is a nonempty subset of the set of perfect equilibria.

Proper Equilibria and Sequential Equilibria

Theorem 4

Suppose that Γ^e is an extensive-form game with perfect recall and τ is a proper equilibrium of the normal representation of Γ^e. Then there exists a vector of belief probabilities π and a behavioral strategy profile σ such that (σ, π) is a sequential equilibrium of Γ^e, and σ is a behavioral representation of τ.

Interpretation

- Just like **perfect equilibrium** selects **sequential equilibria** in the **multiagent** representation, **proper equilibrium** selects **sequential equilibria** in the **normal-form** representation.
- This bridges the gap between:
  - **Strategic-form refinements** (e.g., proper equilibrium), and
  - **Extensive-form rationality** (e.g., sequential equilibrium)

Implication

- Every proper equilibrium of the normal form can be interpreted as a **sequentially rational plan** in the extensive form.
- Ensures consistency between:
  - Mistake-aware normal-form refinements, and
  - On-path and off-path beliefs in dynamic games

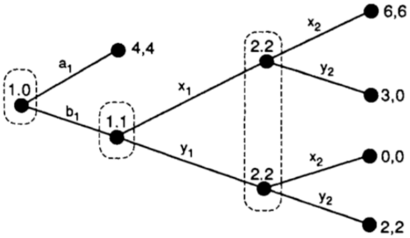
Note

Proper equilibrium is not just a refinement of Nash – it is also a foundation for constructing **sequential equilibria** from the normal form.

Back to Game 4

- Consider **Game 4** and its norml representation:

Game 4



Normal Representation

C_1	C_2: x_2	y_2
a_1x_1	4,4	4,4
a_1y_1	4,4	4,4
b_1x_1	6,6	3,0
b_1y_1	0,0	2,2

Example: ε-Perfect but Not ε-Proper

- Consider the strategy profile:

$$\hat{\sigma} = ((1-\varepsilon)[a_1x_1] + .1\varepsilon[a_1y_1] + .1\varepsilon[b_1x_1] + .8\varepsilon[b_1y_1],\ \varepsilon[x_2] + (1-\varepsilon)[y_2])$$

- This profile is ε-perfect for any $\varepsilon < \frac{1}{3}$
- So as $\varepsilon \to 0$, $\hat{\sigma}$ converges to $([a_1x_1], [y_2])$, a **perfect equilibrium**

Not ε-Proper

- Although $\hat{\sigma}$ is ε-perfect, it is **not ε-proper**
- Why? Consider the strategies b_1x_1 and b_1y_1 for player 1:
  - b_1y_1 is a **worse mistake** than b_1x_1
  - Payoffs:

$$u_1(b_1y_1) = 0\varepsilon + 2(1-\varepsilon) = 2 - 2\varepsilon$$

$$u_1(b_1x_1) = 6\varepsilon + 3(1-\varepsilon) = 3 + 3\varepsilon$$

- But in $\hat{\sigma}$, the ratio of probabilities is:

$$\frac{\hat{\sigma}_1(b_1y_1)}{\hat{\sigma}_1(b_1x_1)} = \frac{0.8\varepsilon}{0.1\varepsilon} = 8$$

- ε-properness requires this ratio $\leq \varepsilon$ – clearly violated

Unique Proper Equilibrium

- For $\varepsilon < 1$, to satisfy ε-properness:
  - Player 1 must assign **less probability** to $b_1 y_1$ than to $b_1 x_1$
  - So x_2 must be a best response for player 2
- Hence, in any proper equilibrium:
  - Player 2 plays x_2
  - Player 1 plays $b_1 x_1$
- The unique proper equilibrium of the strategic-form game is:

$$(b_1 x_1, x_2)$$

Justification by ε-Proper Equilibria

- This proper equilibrium arises as a limit of ε-proper equilibria like:

$$((1-\varepsilon-.5\varepsilon^2)[b_1x_1]+.5\varepsilon^2[b_1y_1]+.5\varepsilon[a_1x_1]+.5\varepsilon[a_1y_1],\ (1-\varepsilon)[x_2]+\varepsilon[y_2])$$

- This sequence respects properness:
  - Mistakes get vanishingly small probability
  - Worse mistakes get **even less** than mild ones

Note

This proper equilibrium corresponds to the **unique sequential equilibrium** of the extensive-form game 4.

Finding a Proper Equilibrium

1. Find the **Nash equilibria** of the normal-form game.
2. For a candidate $\bar{\sigma}$, construct completely mixed strategies
$$\sigma^{\varepsilon} \to \bar{\sigma}.$$
3. Compute and rank the payoffs of all pure strategies against $\sigma^{\varepsilon}_{-i}$.
4. Impose
$$u_i(c_i, \sigma^{\varepsilon}_{-i}) < u_i(e_i, \sigma^{\varepsilon}_{-i}) \Rightarrow \sigma^{\varepsilon}_i(c_i) \leq \varepsilon \sigma^{\varepsilon}_i(e_i).$$
5. Verify these inequalities for sufficiently small ε.

Note

If such a sequence exists, $\bar{\sigma}$ is a **proper equilibrium**.

Required Reading

- **Myerson (1997).**
  - Chapter 5 – 5.1-5.4