International School of Economics at TSU
Microeconomics IV (Game Theory)
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Problem Set 1 – Decision-Theoretic Foundations

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1 – Lotteries and the Formal Setting

Let $\Omega = \{t_1, t_2\}$ and $X = \{\$0, \$4, \$9\}$. Consider the following two acts:

$$f = \begin{bmatrix} & \$0 & \$4 & \$9 \\ t_1 & 0.5 & 0.5 & 0 \\ t_2 & 0 & 0.4 & 0.6 \end{bmatrix} \qquad g = \begin{bmatrix} & \$0 & \$4 & \$9 \\ t_1 & 0.2 & 0.6 & 0.2 \\ t_2 & 0.1 & 0.3 & 0.6 \end{bmatrix}$$

(a) Verify that f and g are valid lotteries, i.e. that both belong to $L = \{h : \Omega \to \Delta(X)\}$.

(b) Write out the compound lottery $k = 0.4f + 0.6g$ explicitly as a matrix. Verify that $k \in L$.

(c) Write out the degenerate lottery [\$4] as a matrix. Does $[\$4] \in L$? Is [\$4] a special case of a compound lottery? Explain.

(d) Let $S = \{t_1\}$. Axiom 2 (Relevance) states that if two acts agree on every state in S, they must be indifferent given S. Construct an act $\tilde{f}$ that differs from f only outside S, and state precisely what Axiom 2 implies about f and $\tilde{f}$.

Problem 2 – The Axioms

(a) Transitivity and strict preference. Suppose $f \succ_S g$ and $g \sim_S h$. Using only the definitions of $\succ_S$ and $\sim_S$ in terms of $\succsim_S$, and Axioms 1A and 1B, prove that $f \succ_S h$. State at each step which axiom you invoke.

(b) Monotonicity (Axiom 3). Suppose $f \succ_S h$ and $\alpha > \beta$, both in $[0, 1]$. Axiom 3 states that $\alpha f + (1-\alpha)h \succ_S \beta f + (1-\beta)h$. Give an intuitive explanation of why a rational agent must satisfy this axiom. What would it mean for a decision-maker's behavior if Axiom 3 were violated?

(c) Continuity (Axiom 4). Explain the role of Axiom 4 in allowing preferences to be represented by real-valued utility numbers. Why is it called a continuity axiom? What type of preferences would violate it? Give a concrete example.

(d) Sure-thing principle (Axiom 6A). State Axiom 6A precisely. Then explain: if $f \succsim_S g$ and $f \succsim_T g$ for disjoint events S and T, why would it be irrational to have $g \succ_{S \cup T} f$? Connect your answer to the notion of a self-consistent theory.

Problem 3 – The Substitution Axioms and the Allais Paradox

Recall the four Allais lotteries (with prizes in millions of dollars):

$$f_1 = 0.10[\$12] + 0.90[\$0], \quad f_2 = 0.11[\$1] + 0.89[\$0],$$

$$f_3 = [\$1], \quad f_4 = 0.10[\$12] + 0.89[\$1] + 0.01[\$0].$$

(a) Compute $0.5f_1 + 0.5f_3$ and $0.5f_2 + 0.5f_4$ explicitly as probability distributions over $\{\$0, \$1, \$12\}$. Verify they are identical. Show that the common preference pattern $f_1 \succ f_2$ and $f_3 \succ f_4$ violates Axiom 5B.

(b) Suppose instead a decision-maker has preferences $f_2 \succ f_1$ and $f_3 \succ f_4$. Is this consistent with the substitution axioms? If so, find the conditions on a utility function u over $\{\$0, \$1, \$12\}$ (normalized so that $u(\$0) = 0$ and $u(\$12) = 1$) that rationalize both preferences simultaneously.

(c) The commitment problem. Suppose $x \succ y$ but $0.5[y] + 0.5[z] \succ [w] \succ 0.5[x] + 0.5[z]$, in violation of Axiom 5B. Consider a sequential decision problem where an agent first decides whether to take prize w or not; if not, a fair coin is tossed, and Heads gives z while Tails gives a choice between x and y. Show that the three possible behavioral assumptions – (i) can commit, (ii) cannot commit but foresees the inconstancy, (iii) cannot commit and cannot foresee – lead to three different outcomes. State which outcome each assumption delivers.

(d) The Allais paradox is sometimes defended as rational behavior under regret theory. In two to three sentences, explain why Myerson's framework rejects this defense – i.e., why the substitution axioms are part of the definition of rationality rather than an additional empirical restriction.

Problem 4 – Calculating Subjective Expected Utility

An entrepreneur chooses among three business strategies. The state space is $\Omega = \{t_H, t_M, t_L\}$ (high, medium, low demand) and the prize space is $X = \{\$0, \$5, \$15, \$30\}$ (profit in thousands). Her subjective beliefs are $p(t_H) = 0.5$, $p(t_M) = 0.3$, $p(t_L) = 0.2$, and her state-independent utility function (normalized with $u(\$0) = 0$ and $u(\$30) = 1$) is:

$$u(\$0) = 0, \quad u(\$5) = 0.3, \quad u(\$15) = 0.7, \quad u(\$30) = 1.$$

The three strategies are:

$$f = \begin{bmatrix} & \$0 & \$5 & \$15 & \$30 \\ t_H & 0 & 0 & 0.2 & 0.8 \\ t_M & 0.1 & 0.3 & 0.6 & 0 \\ t_L & 0.6 & 0.4 & 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} & \$0 & \$5 & \$15 & \$30 \\ t_H & 0 & 0.1 & 0.5 & 0.4 \\ t_M & 0 & 0.4 & 0.6 & 0 \\ t_L & 0.3 & 0.7 & 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} & \$0 & \$5 & \$15 & \$30 \\ t_H & 0 & 1 & 0 & 0 \\ t_M & 0 & 1 & 0 & 0 \\ t_L & 0 & 1 & 0 & 0 \end{bmatrix}$$

(a) Using the formula $E_p(u(f) \mid \Omega) = \sum_{t\in\Omega} p(t) \sum_{x\in X} u(x)\, f(x \mid t)$, compute the subjective expected utility of each strategy and rank them. Which strategy should the entrepreneur choose?

(b) The entrepreneur learns that demand is not low – i.e., she conditions on $S = \{t_H, t_M\}$. Recompute the conditional beliefs using $p(t \mid S) = p(t)/p(S)$ for $t \in S$. Then recompute and re-rank the three conditional expected utilities $E_p(u(f) \mid S)$, $E_p(u(g) \mid S)$, $E_p(u(h) \mid S)$.

(c) Does the ranking change between parts (a) and (b)? Interpret economically: what does it mean when conditioning on new information changes the optimal strategy?

(d) A risk-neutral entrepreneur would use expected monetary value (EMV) $= \sum_t p(t) \sum_x x \cdot f(x \mid t)$ rather than expected utility. Compute EMV for each act under the prior beliefs p. Does the EMV ranking agree with the EU ranking from part (a)? If not, explain why the divergence is consistent with the shape of the utility function given.

Problem 5 – Domination

Consider a decision-maker with choice set $X = \{\alpha, \beta, \gamma, \delta\}$ and state space $\Omega = \{\theta_1, \theta_2, \theta_3\}$. The payoff matrix is:

Decision	θ_1	θ_2	θ_3
α	10	2	4
β	4	4	4
γ	2	6	5
δ	6	3	3

(a) For each decision, find the set of beliefs $p = (p_1, p_2, p_3) \in \Delta(\Omega)$ under which that decision is optimal. Verify in each case that the set of such beliefs is convex.

(b) Is δ strongly dominated? If so, find an explicit randomized strategy $\sigma \in \Delta(X)$ that strongly dominates it, and verify that σ yields a strictly higher payoff than δ in every state.

(c) Is β weakly dominated or strongly dominated? Identify the dominating strategy. Is β ever a best response, and if so under what beliefs?

(d) Theorem 6 states that δ is strongly dominated if and only if no belief $p \in \Delta(\Omega)$ makes δ optimal. Verify this equivalence directly for δ by showing explicitly that no $p \in \Delta(\Omega)$ can make δ optimal – without invoking LP duality. (*Hint*: set up the optimality conditions $E_p[u(\delta)] \geq E_p[u(x)]$ for each $x \in X$ and derive a contradiction.)

Problem 6 – Bayesian Conditional-Probability Systems and Zero-Probability Events

Let the state space be

$$\Omega = \{t_1, t_2, t_3\}.$$

For each $k = 1, 2, ...,$ define a full-support prior $\hat{p}^k \in \Delta^0(\Omega)$ by

$$\hat{p}^k(t_1) = 1 - \frac{1}{k} - \frac{1}{k^2}, \qquad \hat{p}^k(t_2) = \frac{1}{k}, \qquad \hat{p}^k(t_3) = \frac{1}{k^2}.$$

Let p be the limiting conditional-probability system generated by the sequence $\{\hat{p}^k\}_{k=1}^{\infty}$.

(a) Compute the limiting unconditional probabilities

$$p(t_1 \mid \Omega), \qquad p(t_2 \mid \Omega), \qquad p(t_3 \mid \Omega).$$

(b) Compute the limiting conditional probabilities after the event

$$S = \{t_2, t_3\}.$$

That is, compute

$$p(t_2 \mid S), \qquad p(t_3 \mid S).$$

(c) Compute the limiting conditional probabilities after the event

$$T = \{t_1, t_3\}.$$

That is, compute

$$p(t_1 \mid T), \qquad p(t_3 \mid T).$$

(d) Verify Bayes's formula for the nested events

$$R = \{t_3\} \subseteq S = \{t_2, t_3\} \subseteq \Omega.$$

That is, verify that

$$p(R \mid \Omega) = p(R \mid S)p(S \mid \Omega).$$

(e) Explain why this example shows that conditional beliefs after zero-probability events are not arbitrary. Even though $p(S \mid \Omega) = 0$, the conditional belief $p(t_2 \mid S)$ is still well-defined. What determines it?

(f) Think ahead to game theory. In a game, one player's unexpected action may reveal information or force other players to revise their beliefs. Why might it be useful to have a theory that assigns beliefs even after events that were initially considered extremely unlikely or impossible? Explain intuitively.