International School of Economics at TSU
Microeconomics IV (Game Theory)
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Problem Set 2 - Basic Models of Game Theory

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1 We consider a situation involving a **Firm** and a **Worker**.

- The **Worker** can be of two types:
  - **High ability** (prefers to Work if hired)
  - **Low ability** (prefers to Shirk if hired)
- The **Firm** wants to hire the Worker **only if he will Work**.
- The Worker **knows his own type**, but the Firm does **not**.
- The Firm knows that the Worker knows his own type.

Let:

- Probability that the Worker is **High ability**: p
- Probability that the Worker is **Low ability**: $1 - p$

If hired, the Worker chooses whether to **Work** or **Shirk**.

The resulting **payoffs (Firm, Worker)** are:

High Ability Worker

- Hire & Work: $(1, 2)$
- Hire & Shirk: $(0, 1)$
- Not Hire: $(0, 0)$

Low Ability Worker

- Hire & Work: $(1, 1)$
- Hire & Shirk: $(-1, 2)$
- Not Hire: $(0, 0)$

(a) Extensive Form

- Write this situation as an **extensive-form game** with imperfect information.
- Carefully label:
  - The sequence of moves (Nature, Worker, Firm)
  - Information sets
  - Actions and terminal payoffs

(b) Normal Form

- Rewrite the problem in **normal (strategic) form**:

  - Enumerate the **strategy sets** for each player.
  - Construct the **payoff matrix** by considering all strategy combinations.

(c) Multi-Agent Representation

- Represent the situation using a **multi-agent normal form**:
  - Treat each **type of Worker** as a separate agent.
  - List all possible strategies and assign payoffs.

(d) Bayesian Game Representation

- Model the situation as a **Bayesian game**:
  - Define:
    * The set of players
    * The set of types
    * Strategy sets
    * Beliefs
    * Payoff functions
  - Describe how **incomplete information** is handled through beliefs and types.

Problem 2 Suppose that strategy s_1 strictly dominates strategy s_1', and that strategy s_2 strictly dominates strategy s_2'. Let σ_1 be a mixture that attaches positive probability to (only) s_1 and s_2, and σ_1' a mixture that attaches positive probability to (only) s_1' and s_2'. Either prove that σ_1 strictly dominates σ_1', or provide a counterexample.

Problem 3 Consider a game in which player 1 names an amount of money x from the interval $[0, 100]$ to be given to player 2. Player 2 simultaneously names the minimum amount of money y that 2 is willing to accept. If $y > x$, neither player receives anything. If $y \leq x$, player 2 receives x and player 1 receives $100 - x$. Which strategies survive the iterated elimination of weakly dominated strategies? Now consider the same game, except that if $y < x$, 2 receives y and 1 receives $100 - y$. Now which strategies survive iterated weak dominance? Which payoff rule seems to favor player 2? Under which does 2 fare better? (Assume that you can equate monetary payoffs with utilities.)

Problem 4 Consider a three-player extensive-form game. Nature first draws a state $\omega \in \{A, B\}$, each with probability $\frac{1}{2}$. Player 1 observes ω and chooses T or B. Player 2 observes only Player 1's action (not ω) and chooses L or R. Player 3 observes neither ω nor Player 1's action, but does observe Player 2's action, and chooses U or D. Payoffs (u_1, u_2, u_3) at terminal nodes are:

$\omega = A$	L	R
T, U	$(3, 2, 1)$	$(0, 1, 2)$
T, D	$(1, 3, 0)$	$(2, 0, 3)$
B, U	$(0, 1, 3)$	$(3, 2, 0)$
B, D	$(2, 0, 2)$	$(1, 3, 1)$

$\omega = B$	L	R
T, U	$(1, 0, 2)$	$(3, 1, 0)$
T, D	$(2, 3, 1)$	$(0, 2, 3)$
B, U	$(3, 2, 0)$	$(1, 0, 1)$
B, D	$(0, 1, 3)$	$(2, 3, 2)$

(a) Draw the extensive-form game carefully, labeling all nodes, information sets, and terminal payoffs. How many information sets does each player have? Verify that perfect recall holds for all three players.

(b) Write down the strategy sets C_1, C_2, C_3. How many pure strategy profiles are there in total?

(c) Construct the multiagent representation. How many agents are there, and what is each agent's strategy set? Explain why separating a player into temporary agents changes how deviations are checked. What does the multiagent representation help us see, and what does it not determine by itself?

(d) Write the game in Bayesian form Γ^b. Define the type sets, strategy functions, and utility functions formally. Player 2's belief over ω is derived from observing Player 1's action – explain why this belief is in general not equal to the prior $\frac{1}{2}$, and what additional structure would be needed to pin it down.

(e) Suppose we restrict attention to strategies in which Player 1 plays the same action regardless of ω (a **pooling strategy**). Show that under any pooling strategy by Player 1, Player 2's information set is uninformative about ω. Analyze Player 3's situation separately under pooling on T and pooling on B: in each case, identify Player 3's optimal contingent strategy and determine whether a strictly dominant action exists at both information sets.

Problem 5 A government (Player G) must decide whether to **Audit** (A) or **Not Audit** (N) a firm (Player F). The firm has private information about its type: it is either **Compliant** (C) or **Non-Compliant** (NC), with probabilities p and $1 - p$ respectively.

If the government audits, it incurs cost $c > 0$. If it audits a non-compliant firm, it collects fine $f > c$. If it audits a compliant firm, it collects nothing but pays c. A non-compliant firm that is not audited gains $g > 0$.

	A	N
C	$(-c,\ 0)$	$(0,\ 0)$
NC	$(f - c,\ -f)$	$(0,\ g)$

where rows are firm types, columns are government actions, and payoffs are (Government, Firm). The firm first chooses **Disclose** (D) or **Conceal** (K), and the government observes this disclosure decision before deciding whether to audit.

(a) Write this as a Bayesian game Γ^b. Define all components formally.

(b) Draw the extensive form of the extended game. How many information sets does the government have? Write down the government's strategy set.

(c) With $p = \frac{2}{3}$, $c = 1$, $f = 4$, $g = 3$: for what values of μ (the government's posterior belief that the firm is non-compliant, given its observed message) does the government strictly prefer to audit?

(d) Under what conditions does a non-compliant firm prefer to mimic the compliant type by playing D? If both types pool on D, what is the posterior $\mu(D)$? Is auditing rational at this posterior? Does the non-compliant firm strictly prefer to remain at D, or only weakly prefer it? Explain how the answer depends on the government's belief after the message K.

(e) Suppose the compliant type plays D with probability 1, and the non-compliant type plays D with probability $\lambda \in [0, 1]$ and K with probability $1 - \lambda$. The government audits with probability α after D and β after K. Find values of α, β, and λ such that the government's audit decisions are consistent with its posterior beliefs and the non-compliant firm is indifferent between D and K. Under what parameter restriction does an interior value $\lambda \in (0, 1)$ exist?