International School of Economics at TSU
Microeconomics IV (Game Theory)
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Problem Set 3 - Equilibria of strategic-Form Games

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1 Consider the following game:

	A	B	C	D	E
A	$(0,1)$	$(9,0)$	$(2,3)$	$(4,0)$	$(2,6)$
B	$(7,9)$	$(7,3)$	$(1,7)$	$(1,7)$	$(4,5)$
C	$(7,5)$	$(10,10)$	$(3,5)$	$(2,4)$	$(3,3)$

(a) Eliminate any strictly dominating rows and columns. Repeat this process as long as either player has any strictly dominated strategies. Draw the resulting matrix after all strictly dominated strategies are removed.

(b) What is Player 1's security level $\underline{v}_1$? Which strategy (or strategies) guarantee(s) Player 1 at least $\underline{v}_1$?

(c) What is Player 2's security level $\underline{v}_2$? Which strategy (or strategies) guarantee(s) Player 2 at least $\underline{v}_2$?

(d) What are the pure strategy Nash equilibria?

Problem 2 Consider the following variant of Rock–Paper–Scissors:

	Rock	Paper	Scissors
Rock	$(0,0)$	$(-1,1)$	$(4,-4)$
Paper	$(1,-1)$	$(0,0)$	$(-4,4)$
Scissors	$(-4,4)$	$(4,-4)$	$(0,0)$

(a) Is $\left(\frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors},\ \frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors}\right)$ a Nash equilibrium for this game? Explain your answer.

(b) Is $\left(\frac{5}{9}\text{Rock} + \frac{4}{9}\text{Paper},\ \frac{8}{9}\text{Rock} + \frac{1}{9}\text{Scissors}\right)$ a Nash equilibrium for this game? Explain your answer.

(c) Suppose that Player 1 uses the mixed strategy $\frac{5}{9}\text{Rock} + \frac{4}{9}\text{Paper}$ and that Player 2 uses the mixed strategy $\frac{8}{9}\text{Rock} + \frac{1}{9}\text{Scissors}$. What is Player 1's expected payoff? What is Player 2's expected payoff?

Problem 3 Consider the following strategic game:

	A	B
A	$9,3$	$3,1$
B	$8,2$	$6,8$
C	$5,7$	$4,2$
D	$4,5$	$7,9$

Find all Nash equilibria for this game, including pure and mixed strategies.

Problem 4 In the following 3-player game, Player 1 chooses a row (A or B), Player 2 chooses a column (a or b), and Player 3 chooses a matrix (α, β, or γ).

	a	b
A	$0,0,5$	$0,0,0$
B	$2,0,0$	$0,0,0$

α

	a	b
A	$1,2,3$	$0,0,0$
B	$0,0,0$	$1,2,3$

β

	a	b
A	$0,0,0$	$0,0,0$
B	$0,5,0$	$0,0,4$

γ

Find all pure strategy Nash equilibria for this game.

Problem 5 Several strategic settings can be modeled as a tournament, whereby the probability of winning a certain prize not only depends on how much effort you exert, but also on how much effort other participants in the tournament exert. For instance, wars between countries, or R&D competitions between different firms in order to develop a new product, not only depend on a participant's own effort, but on the effort put by its competitors. Let's analyze equilibrium behavior in these settings.

Consider that the benefit that firm 1 obtains from being the first company to launch a new drug is \$36 million. However, the probability of winning this R&D competition against its rival (i.e., being the first to launch the drug) is

$$\frac{x_1}{x_1 + x_2},$$

which increases with this firm's own expenditure on R&D, x_1, relative to total expenditure by both firms, x_1+x_2. Intuitively, this suggests that, while spending more than its rival, i.e., $x_1 > x_2$, increases firm 1's chances of being the winner, the fact that $x_1 > x_2$ does not guarantee that firm 1 will be the winner. That is, there is still some randomness as to which firm will be the first to develop the new drug, e.g., a firm can spend more resources than its rival but be "unlucky" because its laboratory exploits a few weeks before being able to develop the drug.

For simplicity, assume that firms' expenditure cannot exceed 25, i.e., $x_i \in [0, 25]$. The cost is simply x_i, so firm 1's profit function is

$$\pi_1(x_1, x_2) = 36\left(\frac{x_1}{x_1 + x_2}\right) - x_1,$$

and there is an analogous profit function for firm 2:

$$\pi_2(x_1, x_2) = 36\left(\frac{x_2}{x_1 + x_2}\right) - x_2.$$

You can easily check that these profit functions are increasing and concave in a firm's own expenditure. Intuitively, this indicates that, while profits increase in the firm's R&D, the first million dollar is more profitable than the 10th million dollar, e.g., the innovation process is more exhausted.

(a) Find each firm's best-response function.
(b) Find a symmetric Nash equilibrium, i.e., $x_1^* = x_2^* = x^*$.

Problem 6 Let σ^* be an equilibrium in mixed strategies of a strategic-form game. Let s_i and $\hat{s}_i$ be two pure strategies of player i. Assume that both strategies are played with positive probability in the equilibrium: $\sigma_i^*(s_i) > 0$ and $\sigma_i^*(\hat{s}_i) > 0$.

Prove that:

$$U_i(s_i, \sigma_{-i}^*) = U_i(\hat{s}_i, \sigma_{-i}^*)$$

That is, show that any two pure strategies that receive positive probability in a player's mixed strategy must yield the same expected utility against the equilibrium profile of the opponents.

Problem 7 Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form, and let $\widehat{G} = (N, (\widehat{S}_i)_{i\in N}, (u_i)_{i\in N})$ be the game derived from G through the elimination of some of the strategies, namely, $\widehat{S}_i \subseteq S_i$ for each player $i \in N$. Prove that if s^* is an equilibrium in game G, and if $s_i^* \in \widehat{S}_i$ for each player i, then s^* is an equilibrium in the game $\widehat{G}$.