International School of Economics at TSU
Microeconomics IV (Game Theory)
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Problem Set 3 - Equilibria of strategic-Form Games

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1 Consider the following game:

	A	B	C	D	E
A	$(0,1)$	$(9,0)$	$(2,3)$	$(4,0)$	$(2,6)$
B	$(7,9)$	$(7,3)$	$(1,7)$	$(1,7)$	$(4,5)$
C	$(7,5)$	$(10,10)$	$(3,5)$	$(2,4)$	$(3,3)$

(a) Eliminate any strictly dominating rows and columns. Repeat this process as long as either player has any strictly dominated strategies. Draw the resulting matrix after all strictly dominated strategies are removed.

(b) What is Player 1's security level $\underline{v}_1$? Which strategy (or strategies) guarantee(s) Player 1 at least $\underline{v}_1$?

(c) What is Player 2's security level $\underline{v}_2$? Which strategy (or strategies) guarantee(s) Player 2 at least $\underline{v}_2$?

(d) What are the pure strategy Nash equilibria?

Solution:

(a) To start with, column A dominates column D. After we remove column D, row C dominates row A. After we remove row A, column C dominates column E. We are left with the following matrix:

	A	B	C
B	$7,9$	$7,3$	$1,7$
C	$7,5$	$10,10$	$3,5$

(b) Player 1's security level is $\underline{v}_1 = 3$. Strategy C guarantees Player 1 at least 3.

(c) Player 2's security level is $\underline{v}_2 = 5$. Strategies A and C guarantee Player 2 at least 5.

(d) There are two pure strategy Nash equilibria: (B, A) and (C, B).

Problem 2 Consider the following variant of Rock–Paper–Scissors:

	Rock	Paper	Scissors
Rock	$(0, 0)$	$(-1, 1)$	$(4, -4)$
Paper	$(1, -1)$	$(0, 0)$	$(-4, 4)$
Scissors	$(-4, 4)$	$(4, -4)$	$(0, 0)$

(a) Is $\left(\frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors},\ \frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors}\right)$ a Nash equilibrium for this game? Explain your answer.

(b) Is $\left(\frac{5}{9}\text{Rock} + \frac{4}{9}\text{Paper},\ \frac{8}{9}\text{Rock} + \frac{1}{9}\text{Scissors}\right)$ a Nash equilibrium for this game? Explain your answer.

(c) Suppose that Player 1 uses the mixed strategy $\frac{5}{9}\text{Rock} + \frac{4}{9}\text{Paper}$ and that Player 2 uses the mixed strategy $\frac{8}{9}\text{Rock} + \frac{1}{9}\text{Scissors}$. What is Player 1's expected payoff? What is Player 2's expected payoff?

Solution:

(a) Yes, it is a Nash equilibrium. To see this, we start by assuming that Player 1 uses the strategy $\frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors}$, and we determine Player 2's expected value for choosing Rock, Paper, or Scissors:

- If Player 2 chooses Rock, his expected value is:

$$\frac{4}{9}(0) + \frac{4}{9}(-1) + \frac{1}{9}(4) = 0$$

- If Player 2 chooses Paper, his expected value is:

$$\frac{4}{9}(1) + \frac{4}{9}(0) + \frac{1}{9}(-4) = 0$$

- If Player 2 chooses Scissors, his expected value is:

$$\frac{4}{9}(-4) + \frac{4}{9}(4) + \frac{1}{9}(0) = 0$$

Player 2 is indifferent between choosing Rock, Paper, or Scissors, so the mixed strategy $\frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors}$ also gives him the same expected value.

Since the game is symmetric, we'll get the same results if we assume that Player 2 uses the $\frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors}$ — Player 1 will have an expected value of 0 regardless of whether he chooses Rock, Paper, Scissors, or a mixed strategy.

Thus,

$$\left(\frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors},\ \frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors}\right)$$

is a Nash equilibrium.

(b) No, it is not a Nash equilibrium. To see this, we start by assuming that Player 1 uses the strategy $\frac{5}{9}$Rock + $\frac{4}{9}$Paper, and we determine Player 2's expected value if he chooses Rock, Paper, or Scissors:

- If Player 2 chooses Rock, Player 2's expected value is:

$$\frac{5}{9}(0) + \frac{4}{9}(-1) = -\frac{4}{9}$$

- If Player 2 chooses Paper, Player 2's expected value is:

$$\frac{5}{9}(1) + \frac{4}{9}(0) = \frac{5}{9}$$

- If Player 2 chooses Scissors, Player 2's expected value is:

$$\frac{5}{9}(-4) + \frac{4}{9}(4) = -\frac{4}{9}$$

Thus, Player 2's best response is Paper, so he would not use the strategy $\frac{8}{9}$Rock+ $\frac{1}{9}$Scissors, so it is not a Nash equilibrium.

(c) If Player 1 uses the strategy $\frac{5}{9}$Rock + $\frac{4}{9}$Paper, then:

- If Player 2 chooses Rock, Player 2's expected value is:

$$\frac{5}{9}(0) + \frac{4}{9}(-1) = -\frac{4}{9}$$

- If Player 2 chooses Paper, Player 2's expected value is:

$$\frac{5}{9}(1) + \frac{4}{9}(0) = \frac{5}{9}$$

- If Player 2 chooses Scissors, Player 2's expected value is:

$$\frac{5}{9}(-4) + \frac{4}{9}(4) = -\frac{4}{9}$$

Thus, if Player 2 uses the strategy $\frac{8}{9}$Rock + $\frac{1}{9}$Scissors, then Player 2's expected payoff is:

$$\frac{8}{9}\left(-\frac{4}{9}\right) + \frac{1}{9}\left(-\frac{4}{9}\right) = \frac{-36}{81} = -\frac{4}{9}.$$

If Player 2 uses the strategy $\frac{8}{9}$Rock + $\frac{1}{9}$Scissors, then:

- If Player 1 chooses Rock, Player 1's expected value is:

$$\frac{8}{9}(0) + \frac{1}{9}(4) = \frac{4}{9}$$

- If Player 1 chooses Paper, Player 1's expected value is:

$$\frac{8}{9}(1) + \frac{1}{9}(-4) = \frac{4}{9}$$

- If Player 1 chooses Scissors, Player 1's expected value is:

$$\frac{8}{9}(-4) + \frac{1}{9}(0) = -\frac{32}{9}$$

Thus, if Player 1 uses the strategy $\frac{5}{9}\text{Rock} + \frac{4}{9}\text{Paper}$, then Player 1's expected payoff is:

$$\frac{5}{9}\left(\frac{4}{9}\right) + \frac{4}{9}\left(\frac{4}{9}\right) = \frac{36}{81} = \frac{4}{9}.$$

Thus, Player 1's expected payoff is $\frac{4}{9}$ and Player 2's expected payoff is $-\frac{4}{9}$.

Problem 3 Consider the following strategic game:

	A	B
A	$9, 3$	$3, 1$
B	$8, 2$	$6, 8$
C	$5, 7$	$4, 2$
D	$4, 5$	$7, 9$

Find all Nash equilibria for this game, including pure and mixed strategies.

Solution:

(a) First, we note that there are two pure strategy Nash equilibria: (A, A) and (D, B).

Next, we look for mixed strategy Nash equilibria. Suppose that Player 2 uses the mixed strategy $pA + (1-p)B$. Then, Player 1's expected payoffs are:

- If Player 1 chooses A, Player 1's expected value is:

$$p(9) + (1-p)(3) = 3 + 6p$$

- If Player 1 chooses B, Player 1's expected value is:

$$p(8) + (1-p)(6) = 6 + 2p$$

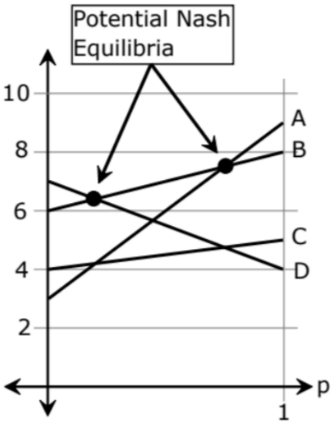
- If Player 1 chooses C, Player 1's expected value is:

$$p(5) + (1-p)(4) = 4 + p$$

- If Player 1 chooses D, Player 1's expected value is:

$$p(4) + (1-p)(7) = 7 - 3p$$

We graph the equations for these lines:



From the graph, we can see that there are two potential mixed strategy Nash equilibria (not counting pure strategy Nash equilibria). One potential Nash equilibrium comes from Player 1 using strategies A and B; the other comes from Player 1 using strategies B and D. We will check both of these separately.

If Player 1 uses strategies A and B, the matrix looks like:

$$\begin{array}{c|cc} & A & B \\ \hline A & 9,3 & 3,1 \\ B & 8,2 & 6,8 \end{array}$$

Suppose that Player 1 uses the mixed strategy $pA + (1-p)B$. Then, Player 2's expected payoffs are:

- If Player 2 chooses A, Player 2's expected value is:

$$p(3) + (1-p)(2) = 2 + p$$

- If Player 2 chooses B, Player 2's expected value is:

$$p(1) + (1-p)(8) = 8 - 7p$$

We set these equal:

$$2 + p = 8 - 7p$$

Solving, we get that $p = 3/4$. Thus, if Player 1 uses the strategy $\frac{3}{4}A + \frac{1}{4}B$, Player 2 will be indifferent between A and B (so that any mixed strategy involving A and B will be a best response for Player 2).

To find the corresponding strategy for Player 2 to use, we set the equations for the lines for A and B equal:

$$3 + 6p = 6 + 2p$$

Solving, we get that $p = 3/4$. Thus, if Player 2 uses the strategy $\frac{3}{4}A + \frac{1}{4}B$, Player 1's best response will be A, B, or any combination of A and B (we can see this by looking at the previous graph). Thus, one mixed strategy Nash equilibrium is:

$$\left(\frac{3}{4}A + \frac{1}{4}B,\ \frac{3}{4}A + \frac{1}{4}B\right)$$

Now, we consider Player 1 using strategies B and D. If Player 1 uses B and D, the matrix looks like:

	A	B
B	$8, 2$	$6, 8$
D	$4, 5$	$7, 9$

Suppose that Player 1 uses the mixed strategy $pB + (1 - p)D$. Then, Player 2's expected payoffs are:

- If Player 2 chooses A, Player 2's expected value is:

$$p(2) + (1 - p)(5) = 5 - 3p$$

- If Player 2 chooses B, Player 2's expected value is:

$$p(8) + (1 - p)(9) = 9 - p$$

We set these equal:

$$5 - 3p = 9 - 4p$$

Solving, we get that $p = 4$. Since $4 > 1$, this does not correspond to a Nash equilibrium. Thus, there is not a Nash equilibrium corresponding to the second point in the graph.

Thus, this game has three Nash equilibria: (A, A), (D, B), and

$$\left(\frac{3}{4}A + \frac{1}{4}B,\ \frac{3}{4}A + \frac{1}{4}B\right)$$

Problem 4 In the following 3-player game, Player 1 chooses a row (A or B), Player 2 chooses a column (a or b), and Player 3 chooses a matrix (α, β, or γ).

$$\underset{\alpha}{\begin{array}{c|cc} & a & b \\ \hline A & 0,0,5 & 0,0,0 \\ B & 2,0,0 & 0,0,0 \end{array}} \qquad \underset{\beta}{\begin{array}{c|cc} & a & b \\ \hline A & 1,2,3 & 0,0,0 \\ B & 0,0,0 & 1,2,3 \end{array}} \qquad \underset{\gamma}{\begin{array}{c|cc} & a & b \\ \hline A & 0,0,0 & 0,0,0 \\ B & 0,5,0 & 0,0,4 \end{array}}$$

Find all pure strategy Nash equilibria for this game.

Solution:

There are four pure strategy Nash equilibria: (A, b, α), (B, a, α), (B, a, γ), and (A, b, γ).

Problem 5 Several strategic settings can be modeled as a tournament, whereby the probability of winning a certain prize not only depends on how much effort you exert, but also on how much effort other participants in the tournament exert. For instance, wars between countries, or R&D competitions between different firms in order to develop a new product, not only depend on a participant's own effort, but on the effort put by its competitors. Let's analyze equilibrium behavior in these settings.

Consider that the benefit that firm 1 obtains from being the first company to launch a new drug is \$36 million. However, the probability of winning this R&D competition against its rival (i.e., being the first to launch the drug) is

$$\frac{x_1}{x_1 + x_2},$$

which increases with this firm's own expenditure on R&D, x_1, relative to total expenditure by both firms, x_1+x_2. Intuitively, this suggests that, while spending more than its rival, i.e., $x_1 > x_2$, increases firm 1's chances of being the winner, the fact that $x_1 > x_2$ does not guarantee that firm 1 will be the winner. That is, there is still some randomness as to which firm will be the first to develop the new drug, e.g., a firm can spend more resources than its rival but be "unlucky" because its laboratory exploits a few weeks before being able to develop the drug.

For simplicity, assume that firms' expenditure cannot exceed 25, i.e., $x_i \in [0, 25]$. The cost is simply x_i, so firm 1's profit function is

$$\pi_1(x_1, x_2) = 36 \left(\frac{x_1}{x_1 + x_2} \right) - x_1,$$

and there is an analogous profit function for firm 2:

$$\pi_2(x_1, x_2) = 36\left(\frac{x_2}{x_1 + x_2}\right) - x_2.$$

You can easily check that these profit functions are increasing and concave in a firm's own expenditure. Intuitively, this indicates that, while profits increase in the firm's R&D, the first million dollar is more profitable than the 10th million dollar, e.g., the innovation process is more exhausted.

(a) Find each firm's best-response function.
(b) Find a symmetric Nash equilibrium, i.e., $x_1^* = x_2^* = x^*$.

Solution:

(a) Firm 1's optimal expenditure is the value of x_1 for which the first derivative of its profit function equals zero. That is,

$$\frac{\partial \pi_1(x_1, x_2)}{\partial x_1} = 36\left[\frac{x_1 + x_2 - x_1}{(x_1 + x_2)^2}\right] - 1 = 0.$$

Rearranging, we find

$$36\left[\frac{x_2}{(x_1 + x_2)^2}\right] - 1 = 0$$

which simplifies to

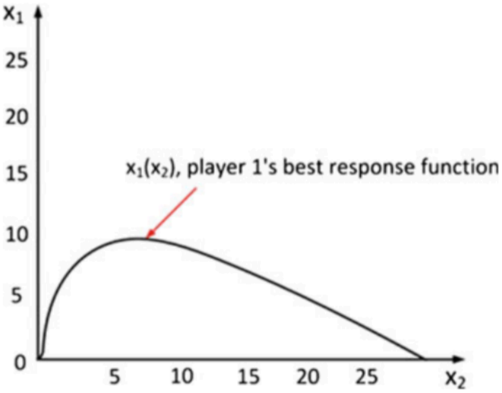
$$36x_2 = (x_1 + x_2)^2$$

and further rearranging

$$6\sqrt{x_2} = x_1 + x_2$$

Solving for x_1, we obtain firm 1's best response function:

$$x_1(x_2) = 6\sqrt{x_2} - x_2$$



It is straightforward to show that, for all values of $x_2 \in [0, 25]$, firm 1's best response also lies in the admissible set $x_1 \in [0, 25]$. In particular, the maximum of BR_1 occurs at $x_2 = 9$ since

$$\frac{\partial BR_1(x_2)}{\partial x_2} = \frac{\partial[6\sqrt{x_2} - x_2]}{\partial x_2} = 3(x_2)^{-1/2} - 1$$

Hence, the point at which this best response function reaches its maximum is that in which its derivative is zero, i.e., $3(x_2)^{-1/2} - 1 = 0$, which yields a value of $x_2 = 9$.

At this point, firm 1's best response function informs us that firm 1 optimally spends $6\sqrt{9} - 9 = 9$. Finally, note that the best response function is concave in its rival's expenditure, x_2, since

$$\frac{\partial BR_1(x_2)}{\partial x_2^2} = -\frac{3}{2}(x_2)^{-3/2} < 0.$$

By symmetry, firm 2's best response function is $x_2(x_1) = 6\sqrt{x_1} - x_1$.

(b) In a symmetric Nash equilibrium $x_1^* = x_2^* = x^*$. Hence, using this property in the best-response functions found in part (a), yields

$$x^* = 6\sqrt{x^*} - x^*$$

Rearranging, we obtain $2x^* = 6\sqrt{x^*}$, and solving for x^*, we find

$$x^* = 9.$$

Hence, the unique symmetric Nash equilibrium has each firm spending 9. As Fig. 2.36 depicts, the points at which the best response function of player 1 and 2 cross each other occur at the 45° line (so the equilibrium is symmetric). In particular, those points are the origin, i.e., $(0, 0)$, but this case is uninteresting since it implies that no firm spends money on R&D, and $(9, 9)$.

Problem 6 Let σ^* be an equilibrium in mixed strategies of a strategic-form game. Let s_i and $\hat{s}_i$ be two pure strategies of player i. Assume that both strategies are played with positive probability in the equilibrium: $\sigma_i^*(s_i) > 0$ and $\sigma_i^*(\hat{s}_i) > 0$.

Prove that:

$$U_i(s_i, \sigma_{-i}^*) = U_i(\hat{s}_i, \sigma_{-i}^*) \tag{1}$$

That is, show that any two pure strategies that receive positive probability in a player's mixed strategy must yield the same expected utility against the equilibrium profile of the opponents.

Solution:

Suppose by contradiction that Equation (1) does not hold. Without loss of generality, suppose that

$$U_i(s_i, \sigma^*_{-i}) > U_i(\hat{s}_i, \sigma^*_{-i}). \tag{2}$$

Let σ'_i be the strategy of player i defined by

$$\sigma'_i(t_i) := \begin{cases} \sigma_i(t_i) & \text{if } t_i \notin \{s_i, \hat{s}_i\}, \\ 0 & \text{if } t_i = \hat{s}_i, \\ \sigma^*_i(s_i) + \sigma^*_i(\hat{s}_i) & \text{if } t_i = s_i. \end{cases}$$

Then

$$\begin{align} U_i(\sigma'_i, \sigma^*_{-i}) &= \sum_{t_i \in S_i} \sigma(t_i) U_i(t_i, \sigma^*_{-i}) \tag{3} \\ &= \sum_{t_i \notin \{s_i, \hat{s}_i\}} \sigma^*(t_i) U_i(t_i, \sigma^*_{-i}) + (\sigma^*(s_i) + \sigma^*(\hat{s}_i)) U_i(s_i, \sigma^*_{-i}) \tag{4} \\ &> \sum_{t_i \notin \{s_i, \hat{s}_i\}} \sigma^*(t_i) U_i(t_i, \sigma^*_{-i}) + \sigma^*(s_i) U_i(s_i, \sigma^*_{-i}) + \sigma^*(\hat{s}_i) U_i(\hat{s}_i, \sigma^*_{-i}) \tag{5} \\ &= \sum_{t_i \in S_i} \sigma^*_i(t_i) U_i(t_i, \sigma^*_{-i}) \tag{6} \\ &= U_i(\sigma^*). \tag{7} \end{align}$$

The equalities in Equation (4) and Equation (6) follow from the definition of σ, and Equation (5) follows from Equation (2). But this contradicts the assumption that σ^* is an equilibrium, because player i can increase his payoff by deviating to strategy σ'_i. This contradiction shows that the assumption that Equation (1) does not hold was wrong, and the theorem therefore holds.

Note: The reason this result holds is simple: if the expected payoff to player i when he plays pure strategy s_i is higher than when he plays $\hat{s}_i$, then he can improve his expected payoff by increasing the probability of playing s_i and decreasing the probability of playing $\hat{s}_i$.

Problem 7 Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form, and let $\widehat{G} = (N, (\widehat{S}_i)_{i \in N}, (u_i)_{i \in N})$ be the game derived from G through the elimination of some of the strategies, namely, $\widehat{S}_i \subseteq S_i$ for each player $i \in N$. Prove that if s^* is an equilibrium in game G, and if $s^*_i \in \widehat{S}_i$ for each player i, then s^* is an equilibrium in the game $\widehat{G}$.

Solution:

Because s^* is an equilibrium of the game G, it follows that for each player i,

$$u_i(s_i, s^*_{-i}) \le u_i(s^*), \quad \forall s_i \in S_i.$$

Because $\widehat{S_i} \subseteq S_i$ for each player $i \in N$, it is the case that

$$u_i(s_i, s^*_{-i}) \le u_i(s^*), \quad \forall s_i \in \widehat{S_i}.$$

Because s^* is a vector of strategies in the game $\widehat{G}$, we conclude that it is an equilibrium of $\widehat{G}$.