International School of Economics at TSU
Microeconomics IV (Game Theory)
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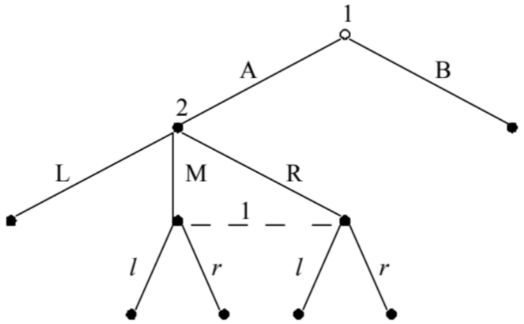
Problem Set 4 - Equilibria of Extensive-Form Games

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1 Consider the following extensive-form game:



a. Let σ_1^* denote player 1's mixed strategy in which she plays (B, r) with probability 0.4, (B, l) with probability 0.1, (A, r) with probability 0.3, and (A, l) with probability 0.2. Find the behavioral strategy of player 1 that is equivalent to σ_1^*. Can you find other mixed strategies that are equivalent to σ_1^*?

b. Compute a behavioral strategy of player 1 that is equivalent to her mixed strategy in which she plays (B, r) with probability 0.7, (B, l) with probability 0.3. Is the behavioral strategy uniquely defined? Explain.

Solution:

a. The behavioral strategy equivalent to σ_1^* is: $\beta(A) = 0.5$, $\beta(l) = 0.4$. This follows from:

$$\begin{aligned}
b_1(A \mid h_1) &= \sigma_1(A,l) + \sigma_1(A,r) = \frac{1}{2} \\
b_1(B \mid h_1) &= \sigma_1(B,l) + \sigma_1(B,r) = \frac{1}{2} \\
b_1(l \mid h_3) &= \frac{\sigma_1(A,l)}{b_1(A \mid h_1)} = \frac{0.2}{0.5} = 0.4 \\
b_1(r \mid h_3) &= \frac{\sigma_1(A,r)}{b_1(A \mid h_1)} = \frac{0.3}{0.5} = 0.6
\end{aligned}$$

Note that, since playing B ends the game, the (l, r) information set h_1 is not reached after B. Hence, any probability distribution such that

$$\sigma_1(B,l) + \sigma_1(B,r) = b_1(B \mid h_1) = \frac{1}{2}$$

and $\sigma_1(A,l) = 0.2$, $\sigma_1(A,r) = 0.3$ will describe a mixed strategy that is equivalent to σ_1^*.

b.

Notice that at the information set h_1:

$$b_1(B \mid h_1) = \sigma_1(B,l) + \sigma_1(B,r) = 1$$

Thus, information set h_3 is never reached, and (l, r) are never played. Therefore, any behavioral strategy that assigns $b_1(B \mid h_1) = 1$ and $b_1(l \mid h_3) = q$, $b_1(r \mid h_3) = 1 - q$ for any $q \in [0, 1]$ is equivalent to the mixed strategy given in the question.

This implies that there is no unique behavioral strategy representation.

Problem 2 Consider the following game. Player 1 first decides between two games A or B, then the game chosen by player 1 is played simultaneously. Player 1 knows which game is played, but player 2 does not know.

Game A

	L	R
U	(1, 0)	(0, 1)
D	(-1, 2)	(2, 0)

Game B

	L	R
U	(4, 3)	(-3, 2)
D	(1, 0)	(0, 0)

a. Find an arbitrary mixed strategy Nash equilibrium of this game in which each pure strategy is played with positive probability.

b. For the mixed strategy equilibrium you obtained in part (a), give an equivalent presentation in behavioral strategies.

Solution:

a. The sets of (pure) strategies are:

$$S_1 = \{AUU, AUD, ADU, ADD, BUU, BUD, BDU, BDD\}$$

$$S_2 = \{L, R\}$$

Mixed Strategy Profile

Now, define the mixed strategies:

$$\begin{aligned}
\sigma_1^*(AUU) = \sigma_1^*(AUD) &= \frac{p}{2} \\
\sigma_1^*(ADU) = \sigma_1^*(ADD) &= \frac{q}{2} \\
\sigma_1^*(BUU) = \sigma_1^*(BDU) &= \frac{r}{2} \\
\sigma_1^*(BUD) = \sigma_1^*(BDD) &= \frac{1-p-q-r}{2} \\
\sigma_2^*(L) = x, \quad \sigma_2^*(R) &= 1-x
\end{aligned}$$

Note: We assume payoff-equivalent strategies are assigned equal probability to simplify the calculations.

Indifference Condition for Player 2

Player 2's expected payoffs:

$$\begin{aligned}
EU_2(L \mid \sigma_1^*) &= 2q + 3r \\
EU_2(R \mid \sigma_1^*) &= p + 2r
\end{aligned}$$

To mix, player 2 must be indifferent between L and R:

$$EU_2(L \mid \sigma_1^*) = EU_2(R \mid \sigma_1^*) \Rightarrow 2q = p - r \tag{1}$$

Indifference Condition for Player 1

Player 1's expected payoffs:

$$\begin{aligned}
EU_1(AUU, \sigma_2^*) = EU_1(AUD, \sigma_2^*) &= x \\
EU_1(ADU, \sigma_2^*) = EU_1(ADD, \sigma_2^*) &= 2 - 3x \\
EU_1(BUU, \sigma_2^*) = EU_1(BDU, \sigma_2^*) &= 7x - 3 \\
EU_1(BUD, \sigma_2^*) = EU_1(BDD, \sigma_2^*) &= x
\end{aligned}$$

To assign positive probability to all pure strategies, player 1 must satisfy:

$$EU_1(AUU,\sigma_2^*) = EU_1(ADU,\sigma_2^*) = EU_1(BUU,\sigma_2^*) = EU_1(BUD,\sigma_2^*) \Rightarrow x = \frac{1}{2} \tag{2}$$

Example Mixed Strategy Nash Equilibrium

Any mixed strategy satisfying both equations (1) and (2) is a mixed strategy NE.

For example, the following profile constitutes a NE:

$$\begin{aligned}
\sigma_1^*(AUU) = \sigma_1^*(AUD) &= 0.2 \\
\sigma_1^*(ADU) = \sigma_1^*(ADD) &= 0.05 \\
\sigma_1^*(BUU) = \sigma_1^*(BDU) &= 0.1 \\
\sigma_1^*(BUD) = \sigma_1^*(BDD) &= 0.15 \\
\sigma_2^*(L) = \sigma_2^*(R) &= 0.5
\end{aligned}$$

b. The equivalent behavioral strategies are:

$$\beta_1(A) = 0.5, \quad \beta_1(U \mid A) = 0.8, \quad \beta_1(U \mid B) = 0.4, \quad \beta_2(L) = 0.5.$$

Problem 3 Consider the following three-player sequential game. In the first stage, P1 can either play L, ending the game with payoffs $(6,0,6)$, or play R, which gives the move to P2. P2 can then play either r, ending the game with payoffs $(8,6,8)$, or play ℓ, thereby moving the game to the third stage. In stage 3, P1 and P3 (but not P2) play a simultaneous-move coordination game: they each choose F or G. If their choices differ, they each receive 7 and P2 gets 10; if their choices coincide, all three players get 0.

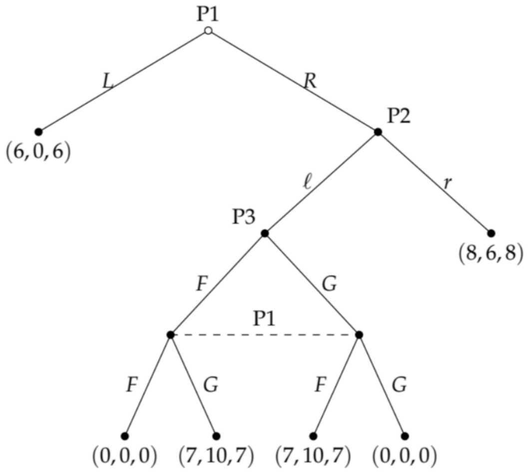
a. Draw the extensive form game in question.

b. Find ALL subgame perfect equilibria in this game.

c. Take note of P1's choice in stage 1 in all of the subgame perfect equilibria. Discuss carefully whether it could be rational for P1 to choose a different action in stage 1.

Solution:

a.



b. Recall that a SPE induces a NE in every subgame. There are three subgames in this game (each stage initiates a subgame).

In the smallest subgame, the coordination game in stage 3, there are three NEs. Two of these NEs are in pure strategies: (i) P1 chooses F and P3 chooses G, and (ii) P3 chooses F and P1 chooses G; both equilibria lead to the payoff vector $(7, 10, 7)$. There is also a mixed strategy NE, in which both P1 and P3 randomize between F and G with equal probability. This mixed NE leads to an expected payoff vector $(3.5, 5, 3.5)$.

In stage 2, P2 chooses ℓ if the equilibrium in stage 3 is either of the two pure strategy NEs; otherwise, he chooses r if the equilibrium in stage 3 is in mixed strategies.

In stage 1, P1 chooses R, anticipating a payoff of either 7 (when she can successfully coordinate with P3 in stage 3) or 8 (when she fails to coordinate with P3 and hence playing the mixed NE in stage 3).

To summarize, there are three SPEs: two in pure strategies,
$(RF; \ell; G)$ and $(RG; \ell; F)$, and one involving mixing
$(R(\frac{1}{2}F + \frac{1}{2}G); r; (\frac{1}{2}F + \frac{1}{2}G))$.

c. A subgame perfect equilibrium requires not only that P1 and P2 anticipate NEs in all subgames, but also that the NEs they anticipate are the *same.* This is a point that Rabin (1988) makes in critiquing the notion of subgame perfection.

In our example, suppose P1 anticipates the mixed strategy NE to be played in stage 3 (if reached). But she also believes that P2 would expect the stage 3 outcome to be one of the pure strategy NEs (and hence P2 would choose ℓ). Then P1 would choose L to secure a payoff of 6, as opposed to the expected payoff of 3.5 if he chooses R as dictated by her equilibrium strategy in any SPE.

Problem 4 Consider the following three-player normal form game, where $S_1 = \{U, D\}$, $S_2 = \{L, R\}$, and $S_3 = \{X, Y\}$:

X

	L	R
U	$(1, 1, 1)$	$(1, 0, 1)$
D	$(1, 1, 1)$	$(0, 0, 1)$

Y

	L	R
U	$(1, 1, 0)$	$(0, 0, 0)$
D	$(0, 1, 0)$	$(1, 0, 0)$

a. Perform iterated deletion of weakly dominated strategies. What are the strategies that survive this process?

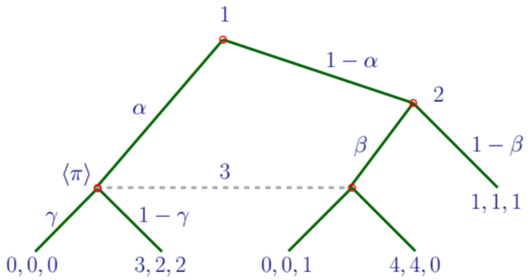
b. Find all Nash equilibria (both pure and mixed). Does any of the Nash equilibrium involve weakly dominated strategies?

Solution:

a. L and X are the strictly dominant strategy for players 2 and 3, respectively. However, there is no domination between U and D, either before or after removing the other players' dominated strategies. So the strategies that survive IDWDS are $\{\{U, D\}, \{L\}, \{X\}\}$.

b. In a NE, both players 2 and 3 would play their respective dominant strategy. Player 1 is then indifferent between U and D. So the NEs of this game can be described by $(pU + (1-p)D, L, X)$, for any $p \in [0, 1]$. None of the NEs involves weakly dominated strategies.

Problem 5 Find the sequential equilibria of the three-player game represented below. Actions are always left or right. For simplicity, we write behavioral probabilities:



- Let α, β, and γ denote the probabilities with which players 1, 2, and 3 play **Left** at their respective information sets.
- Let π denote player 3's belief that they are at the **left** node of their information set.

We are looking for all combinations of strategies (α, β, γ) and belief π that are **sequentially rational** and **consistent**.

Solution:

First we show that it cannot be $0 < \alpha < 1$ (that is the first player cannot mix). By contradiction, suppose it were so. Then 1 must be indifferent between L and R so that $3(1-\gamma)$ must equal $1-\beta+4\beta(1-\gamma)$, that is $2-3\gamma = \beta(3-4\gamma)$. Also, given $1-\alpha > 0$ it is $L \succeq_2 R$ iff $4(1-\gamma) \geq 1$ that is $\gamma \leq 3/4$. But then as you can easily check for no value of γ can the equality $2-3\gamma = \beta(3-4\gamma)$ be satisfied (for example if $\gamma < 3/4$ then $\beta = 1$ so...). Whence either $\alpha = 0$ or $\alpha = 1$.

Suppose $\alpha = 1$. Then $\gamma = 0$; but then it should be $\beta < 1$ (otherwise 1 would deviate to R and get 4) which implies $R \succeq_2 L$ that is $\gamma \geq 3/4$ – contradiction.

Therefore in equilibrium it must be $\alpha = 0$. Given this, if $\beta > 0$ then $\gamma = 1$ which implies $\beta = 0$ – another contradiction. Hence also $\beta = 0$.
To sustain the play R, R we need $R \succeq_{1,2} L$ and this is possible if γ is such that $\gamma \geq 3/4$. That is we need $L \succeq_3 R$ which means $1-\pi \geq 2\pi$ or $\pi \leq 1/3$.

The conclusion so far is that the sequentially rational systems are strategies (α, β, γ) with $\alpha = \beta = 0$ and $\gamma \geq 3/4$ and beliefs $\pi \leq 1/3$.

To finish we show that these π's are consistent. We need sequences of fully mixed $(\alpha_n, \beta_n, \gamma_n) \to (\alpha, \beta, \gamma)$ and $\pi_n \to \pi$ with

$$\pi_n = \frac{\alpha}{\alpha + (1-\alpha)\beta}$$

Note that $\pi \leq 1/3$ is equivalent to $(1-\pi)/\pi \geq 2$ so it suffices to have $(1-\pi_n)/\pi_n = (1-\alpha_n)\beta_n/\alpha_n \geq 2$.
To this end we may take $\gamma_n = \gamma$, $0 < \alpha_n \to 0$ and $\beta_n = c\alpha_n/(1-\alpha_n)$ with any $c \geq 2$ (of course $\beta_n \to 0$).