International School of Economics at TSU
Microeconomics IV (Game Theory)
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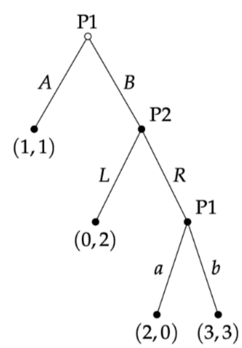
Problem Set 5 - Perfect and Proper Equilibrium. Bayesian Nash Equilibrium. Perfect Bayesian Nash Equilibrium.

Instructions: You are encouraged to solve the problems before the recitation. Additionally, you are encouraged to work in groups. It is **not mandatory** to submit solutions unless stated otherwise. However, if you would like to share your solution, I would be happy to review it.

Problem 1 Problem 1 Consider the following extensive form game



a. Find all subgame perfect equilibria of this game.

Solution

Using backward induction, we can find (Bb, R) to be the unique SPE.

b. Convert the game to its normal form. Find all Nash equilibria in pure strategies.

Solution

The normal form is

	P1 Aa	Ab	Ba	Bb
P2 L	$1, 1$	$1, 1$	$2, 0$	$2, 0$
R	$1, 1$	$1, 1$	$0, 2$	$3, 3$

In addition to the SPE (Bb, R), there are two more NEs that are not subgame perfect: (Aa, L) and (Ab, L).

c. Verify whether these Nash equilibria are trembling hand perfect.

Solution:

We first argue that the SPE (Bb, R) is THP. Suppose for P2, the sequence of totally mixed strategy is

$$\sigma_2^k = \varepsilon^k L + (1-\varepsilon^k)R, \quad \text{for } k \in \mathbb{N} \text{ and some } \varepsilon \in (0,1).$$

Clearly, $\sigma_2^k \to R$ as $k \to \infty$. We need to show that Bb is a best response to σ_2^k, for all $k \in \mathbb{N}$. Observe that

$$\begin{aligned} u_1(Aa, \sigma_2^k) &= 1 \\ u_1(Ab, \sigma_2^k) &= 1 \\ u_1(Ba, \sigma_2^k) &= 2(1-\varepsilon^k) \\ u_1(Bb, \sigma_2^k) &= 3(1-\varepsilon^k) \end{aligned}$$

Thus for Bb to be a best response to σ_2^k, it suffices to let $3(1-\varepsilon) \geq 1$, or $\varepsilon \leq \frac{2}{3}$. We also need R to be a best response to P1's sequence of completely mixed strategy σ_1^k. Suppose

$$\sigma_1^k = \varepsilon^k Aa + \varepsilon^k Ab + \varepsilon^k Ba + (1-3\varepsilon^k)Bb,$$

where $\varepsilon^k \in (0,1)$ and $3\varepsilon^k < 1$ for each k. Clearly, $\sigma_1^k \to Bb$ as $k \to \infty$.
We need to find ε^k such that

$$u_2(\sigma_1^k, L) \leq u_2(\sigma_1^k, R)$$

That is,

$$\varepsilon^k + \varepsilon^k + 2\varepsilon^k + 2(1-3\varepsilon^k) \leq \varepsilon^k + \varepsilon^k + 3(1-3\varepsilon^k)$$

It suffices to pick $\varepsilon \leq \frac{1}{5}$. Hence, there exists a sequence of completely mixed strategy profiles $\sigma^k = (\sigma_1^k, \sigma_2^k)$ as described above, such that $\sigma^k \to (Bb, R)$ and each player's equilibrium strategy is a best response to the other player's trembled strategy in σ^k for every k. Therefore, (Bb, R) is a THP NE.

Next, we argue that (Aa, L) (and (Ab, L)) is THP. Let $\sigma_2^k = (1-\varepsilon^k)L + \varepsilon^k R$, and we want to find an $\varepsilon \in (0,1)$ such that Aa is a best response to σ_2^k. Observe that

$$\begin{aligned} u_1(Aa, \sigma_2^k) &= 1 \\ u_1(Ab, \sigma_2^k) &= 1 \\ u_1(Ba, \sigma_2^k) &= 2\varepsilon^k \\ u_1(Bb, \sigma_2^k) &= 3\varepsilon^k \end{aligned}$$

It suffices to pick $\varepsilon \leq \frac{1}{3}$. We also need to make L a best response to

$$\sigma_1^k = (1 - 3\varepsilon^k)Aa + \varepsilon^k Ab + \varepsilon^k Ba + \varepsilon^k Bb.$$

In fact, given this σ_1^k, any $\varepsilon \in (0, 1)$ will make L a best response:

$$(1 - 3\varepsilon^k) + \varepsilon^k + 2\varepsilon^k + 2\varepsilon^k > (1 - 3\varepsilon^k) + \varepsilon^k + 3\varepsilon^k, \quad \forall \varepsilon \in (0, 1), \forall k \in \mathbb{N}.$$

Hence, we conclude that (Aa, L) is a THP NE. The argument for (Ab, L) being THP is similar.

This example shows that trembling hand perfection, as we have studied in class (which is defined based on normal form games), is a weaker criterion than subgame perfection in ruling out implausible NEs. A stronger version of THP is the **extensive form trembling hand perfection**, which requires a player's strategy to be robust not only to other players' mistakes, but also to one's own mistakes made at different information sets.
See pp. 299–301 of MWG for a discussion.

Problem 2 Consider the following game:

	L	R
T	$3, 1$	$0, 0$
M	$0, 0$	$1, 5$
B	$2, 2$	$2, 2$

a. Find the perfect, strictly perfect[1], and proper equilibria of this game.

b. Now add a fourth pure strategy for player 1 to this game, whose payoff is equivalent to that of a mixture that places probability $1/4$ on T and $3/4$ on B.

Notice that in a sense, we are adding nothing new to the game, since player 1 could already achieve the payoffs provided by this strategy by playing the appropriate mixture.

Find the Nash, perfect, strictly perfect, and proper equilibria of this new game.

How do you interpret your results? For example, in your view, is the newly added strategy redundant, or does it add something new to the game?

If redundant, how do you interpret its effect? If not redundant, what new does it add to the game?

In particular, if it is not redundant, how do we know when a model of this sort should include this strategy (and, presumably, many other strategies of this type) and when it should not?

[1] A **strictly perfect equilibrium** is a trembling-hand perfect equilibrium in which each player's strategy is the *unique* best response to the trembled strategies of the opponents. That is, no other strategy yields the same expected payoff, even in the limit of vanishing trembles.

Solution

Consider the original game. First, we must identify the Nash equilibria of the game and determine which ones satisfy the refinements. It is straightforward to verify that the Nash equilibria are: (T, L), (B, R), and player 1 plays B, player 2 randomizes between L and R with $\sigma_2(R) \geq \frac{1}{3}$.

First, notice that (T, L) is a strict equilibrium, and so satisfies all of the refinements: perfect, strictly perfect, and proper.

What about the strategies that have player 1 choosing B? We can notice that they are all perfect: A tremble where Player 1 plays T w.p. $\frac{5\varepsilon}{6}$, M w.p. $\frac{\varepsilon}{6}$ and B w.p. $1 - \varepsilon$ makes Player 2 indifferent between L and R and so willing to randomize or choose R. Any tremble where Player 2 plays R w.p. at least $\frac{1}{3}$ makes 1's best response to choose B.

Are the equilibria proper? First notice that (B, R) is proper: If 2 is playing R then 1 strictly prefers M to T, so in a proper equilibrium 1 must tremble predominantly to M, in which case R is in fact the best response. The equilibrium where 2 plays R w.p. $\frac{3}{4}$ and L w.p. $\frac{1}{4}$ is also proper since given that strategy player 1 is indifferent between T and M so the above trembles that show that the equilibrium is perfect are also consistent with a proper equilibrium. Any other mixture is not proper since if $\sigma_2^\varepsilon(R) > \frac{3}{4}$, player 1 will predominantly tremble on M making 2 strictly prefer R, and similarly if $\sigma_2^\varepsilon(R) < \frac{3}{4}$, player 1 will tremble predominantly on T making 2 prefer L.

Finally, notice that (T, L) is the only strictly perfect equilibrium since a tremble that puts more than five times as much weight on T as M makes L the unique best response for player 2.

Now suppose we add the new strategy that is equivalent to 1/4 on T and 3/4 on B. So now the game becomes

	L	R
T	$3, 1$	$0, 0$
M	$0, 0$	$1, 5$
B	$2, 2$	$2, 2$
$1/4T + 3/4B$	$2.25, 1.75$	$1.5, 1.5$

The Nash equilibria are unchanged and all of the equilibria remain perfect: (T, L) is still a strict equilibrium, and if 1 trembles by playing B w.p. $1 - \varepsilon$, T w.p. $\frac{4\varepsilon}{9}$, $1/4T + 3/4B$ w.p. $\frac{4\varepsilon}{9}$ and M w.p. $\frac{\varepsilon}{9}$ makes player 2 indifferent between L and R. However, notice that (T, L) is the only proper equilibrium: Take any equilibrium where player 1 is playing B. Since M is strictly dominated for player 1, for the equilibrium to be proper there must be trembles satisfying

$$\sigma_1^\varepsilon(M) < \varepsilon \sigma_1^\varepsilon(1/4T + 3/4B)$$

that converge to the equilibrium. But for ε sufficiently small, L is the unique best response for player 2, so the equilibrium cannot be proper (an equilibrium where 1 plays B requires that 2 plays L w.p. no higher than $\frac{2}{3}$). Since (T, L) is the only proper equilibrium it is also the only strictly perfect equilibrium.

We see from this example that proper equilibria are also not "stable": adding the strategy $1/4T + 3/4B$ did not change the available strategies in the game (1 already had the ability to play $1/4T + 3/4B$) however it changed the set of proper equilibria.

Problem 3 Consider the following Bayesian game.

- Nature selects Game 1 with probability 1/3, Game 2 with probability 1/3, and Game 3 with probability 1/3.
- Player I learns whether Nature has selected Game 1 or not; Player II learns whether Nature has selected Game 2 or not.
- Players I and II simultaneously choose their actions: player I either T or B, and player II either L or R.
- Payoffs are given by the game selected by Nature.

Game 1

	L	R
T	(0,0)	(6,-1)
B	(-1,6)	(4,4)

Game 2

	L	R
T	(1,3)	(0,0)
B	(0,0)	(3,1)

Game 3

	L	R
T	(2,-2)	(-2,2)
B	(-2,2)	(2,-2)

All of this is common knowledge. **Find all the Bayesian Nash equilibria.**

Solution:

- There are 2 players: player I and player II;
- Type spaces: $T_1 = \{\{1\}, \{2, 3\}\}$, and $T_2 = \{\{1, 3\}, \{2\}\}$;
- Beliefs: player I's belief on player II's types: 2/3 on $\{1, 3\}$ and 1/3 on $\{2\}$; player II's belief on player I's types: 1/3 on $\{1\}$ and 2/3 on $\{2, 3\}$;
- Action spaces: $A_1 = \{T, B\}$, and $A_2 = \{L, R\}$;
- Strategy spaces: $S_1 = \{TT, TB, BT, BB\}$, and $S_2 = \{LL, LR, RL, RR\}$.

Now we will find the best-response correspondence for each player and each associated type: let a_1 and a_2 be player I's actions in Game 1, and Games 2 and 3, respectively, b_1 and b_2 player II's actions in Games 1 and 3, and Game 2, respectively.

- If Game 1 is drawn, then player I's best-response correspondence is

$$a_1^*(b_1) = \begin{cases} T, & \text{if } b_1 = L; \\ T, & \text{if } b_1 = R. \end{cases}$$

- If Game 1 is not drawn, then by considering the expected payoff, player I's best-response correspondence is

$$a_2^*(b_1 b_2) = \begin{cases} T, & \text{if } b_1 b_2 = LL; \\ T, & \text{if } b_1 b_2 = LR; \\ B, & \text{if } b_1 b_2 = RL; \\ B, & \text{if } b_1 b_2 = RR. \end{cases}$$

- If Game 2 is drawn, then player II's best-response correspondence is

$$b_2^*(a_2) = \begin{cases} L, & \text{if } a_2 = T; \\ R, & \text{if } a_2 = B. \end{cases}$$

- If Game 2 is not drawn, then by considering the expected payoff, player II's best-response correspondence is

$$b_1^*(a_1 a_2) = \begin{cases} R, & \text{if } a_1 a_2 = TT; \\ L, & \text{if } a_1 a_2 = TB; \\ R, & \text{if } a_1 a_2 = BT; \\ L, & \text{if } a_1 a_2 = BB. \end{cases}$$

Therefore, by definition, we will get all the Bayesian Nash equilibria: (TT, RL) and (TB, LR). The reason is as follows:

- If player I chooses TT, then player II should choose RL; on the other hand, TT is not a best response for RL. So there is no Bayesian Nash equilibrium when player I chooses TT.
- If player I chooses TB, then player II should choose LR; on the other hand, TB is a not best response for LR. So there is no Bayesian Nash equilibrium when player I chooses TB.
- If player I chooses BT, then player II should choose RL; on the other hand, BT is not a best response for RL. So there is no Bayesian Nash equilibrium when player I chooses BT.
- If player I chooses BB, then player II should choose LR; on the other hand, BB is not a best response for LR. So there is no Bayesian Nash equilibrium when player I chooses BB.

Problem 4 Consider the following Bayesian game. Player 1's type t_1 is drawn from a uniform distribution on the interval from 0 to 1, and payoffs (u_1, u_2) depend on Player 1's type as follows, where ε is a number between 0 and 1 (say $\varepsilon = 0.1$):

	L	R
T	$(\varepsilon t_1, 0)$	$(\varepsilon t_1, -1)$
B	$(1, 0)$	$(-1, 3)$

Derive the Bayesian Nash equilibrium of the given game.

Solution:

Player 1's payoffs satisfy increasing differences, so player 1 should use a cutoff strategy:
- Choose T if $t_1 > \theta_1$,
- Choose B if $t_1 < \theta_1$,

where $\theta_1 \in (0, 1)$.

Then player 2 believes that the probability of player 1 choosing T is:

$$\Pr(t_1 > \theta_1) = 1 - \theta_1.$$

You can easily verify that there is **no equilibrium** where player 2 is sure to choose either L or R.

For player 2 to be willing to randomize between L and R, both L and R must give her the same expected payoff:

$$0 = (-1)(1 - \theta_1) + (3)\theta_1 \implies \theta_1 = 0.25.$$

So in equilibrium, player 1 must use the strategy: - Do T if $t_1 > 0.25$, - Do B if $t_1 < 0.25$.

For player 1 to be willing to implement this strategy, he must be **indifferent** between T and B when his type is exactly $t_1 = \theta_1 = 0.25$.
Let q denote the probability of player 2 choosing L.
Then the indifference condition for player 1 at $t_1 = \theta_1$ is:

$$\varepsilon\theta_1 = q(1) + (1 - q)(-1) \implies q = \frac{1 + \varepsilon\theta_1}{2} = \frac{1 + 0.25\varepsilon}{2}.$$

As $\varepsilon \to 0$, we have $q \to 0.5$, so player 2 mixes equally between L and R.

Problem 5 In the following extensive-form games, derive the normal-form game and **find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.**

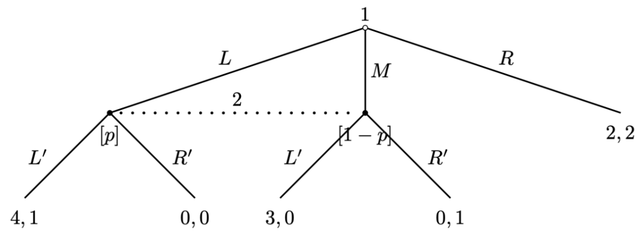


Figure 1: Game 1

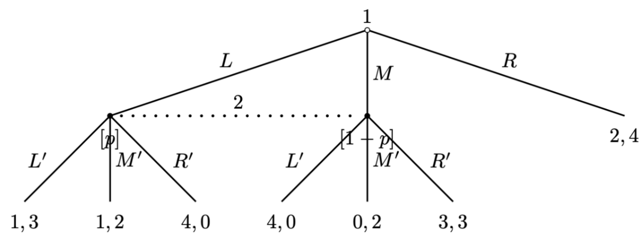


Figure 2: Game 2

Solution:

(a) Game 1. Normal-form representation is as follows:

- $S_1 = \{L, M, R\}$, $S_2 = \{L', R'\}$.
- Payoff table:

	L'	R'
L	$4, 1$	$0, 0$
M	$3, 0$	$0, 1$
R	$2, 2$	$2, 2$

(i) There are two pure-strategy Nash equilibria (L, L') and (R, R').

(ii) Since there is no subgame, every Nash equilibrium is subgame-perfect, and hence (L, L') and (R, R') are all the subgame-perfect Nash equilibria.

(iii) To check whether (L, L') is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 2's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support L' to be a best response for Player 2, we should take $p \geq \frac{1}{2}$.

- Requirement 3: Since Player 1 chooses L, by Bayes' rule, Player 2's belief should be $(1, 0)$, that is, $p = 1$.
- Requirement 4: No information set is off the path, so Requirement 4 gives no restriction on p.

Hence (L, L') with $p = 1$ is a perfect Bayesian equilibrium.

To check whether (R, R') is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 2's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support R' to be a best response for Player 2, we should take $p \leq \frac{1}{2}$.
- Requirement 3: No nontrivial information set is on the path, so Requirement 3 gives no restriction on p.
- Requirement 4: Since Player 1 chooses R, Player 2's information set is off the path, so p could be arbitrary.

Hence (R, R') with $p \leq \frac{1}{2}$ is a perfect Bayesian equilibrium.

(b) Game 2. Normal-form representation is as follows:

- $S_1 = \{L, M, R\}$, $S_2 = \{L', M', R'\}$.
- Payoff table:

	L'	M'	R'
L	$1, 3$	$1, 2$	$4, 0$
M	$4, 0$	$0, 2$	$3, 3$
R	$2, 4$	$2, 4$	$2, 4$

(i) (R, M') is the unique pure-strategy Nash equilibrium.

(ii) Since there is no subgame, every Nash equilibrium is subgame-perfect, and hence (R, M') is the unique subgame-perfect Nash equilibrium.

(iii) To check whether (R, M') is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2, 3 and 4.

- Requirement 1: For Player 2's information set, assign probability p on the left decision node, and $1 - p$ on the right decision node.
- Requirement 2: To support M' to be a best response for Player 2, we should take $p \in [\frac{1}{3}, \frac{2}{3}]$.
- Requirement 3: No nontrivial information set is on the path, so Requirement 3 gives no restriction on p.
- Requirement 4: Since Player 1 chooses R, Player 2's information set is off the path, so p could be arbitrary.

Hence (R, M') with $p \in [\frac{1}{3}, \frac{2}{3}]$ is the unique perfect Bayesian equilibrium.

Problem 6 Find all perfect Bayesian equilibria in the following signaling game.

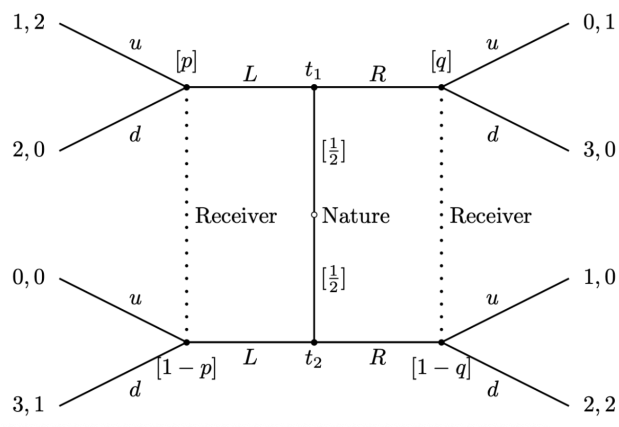


Figure 3: Game 3

Solution:

The normal-form representation is as follows:

- $T = \{t_1, t_2\}$, $M = \{L, R\}$, $A = \{u, d\}$.
- Payoff table:

	Receiver			
	uu	ud	du	dd
LL	$1/2, 1$	$1/2, 1$	$5/2, 1/2$	$5/2, 1/2$
LR	$1, 1$	$3/2, 2$	$3/2, 0$	$2, 1$
RL	$0, 1/2$	$3/2, 0$	$3/2, 1$	$3, 1/2$
RR	$1/2, 1/2$	$5/2, 1$	$1/2, 1/2$	$5/2, 1$

For example,

$$U(RL, du) = \text{Prob}(t_1)U(R, u \mid t_1) + \text{Prob}(t_2)U(L, d \mid t_2) \quad = \frac{1}{2}(0, 1) + \frac{1}{2}(3, 1) = (3/2, 1).$$

There is the unique Nash equilibrium (RR, ud), which is also the unique subgame-perfect Nash equilibrium since there is no subgame.

To check whether (RR, ud) is a perfect Bayesian equilibrium, we need only to find beliefs, satisfying Requirements 1, 2S, 2R and 3.

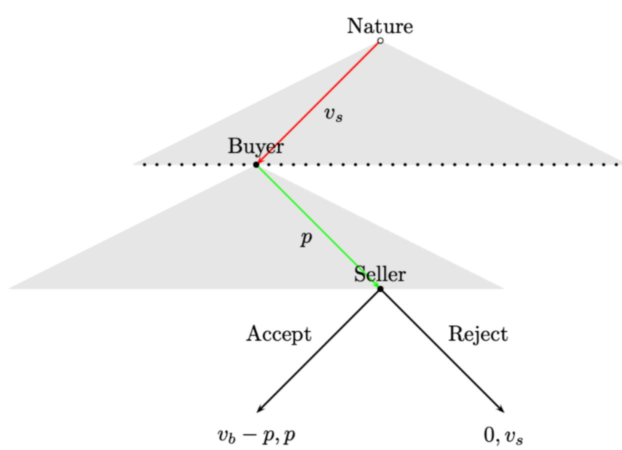
- Requirement 1: Assume the probability distributions on left and right information set are $(p, 1-p)$ and $(q, 1-q)$, respectively, displayed in the figure.
- Requirement 2S: Holds automatically. (since (RR, ud) is a Nash equilibrium)
- Requirement 2R: To support ud (u when Sender chooses L, d when Sender chooses R) to be a best response for Receiver, we should take $p \geq 1/3$ and $q \leq 2/3$.
- Requirement 3: Since Sender chooses RR, Bayes' rule implies p could be arbitrary, and $q = 1/2$.

Hence (RR, ud) with $p \geq 1/3$ and $q = 1/2$ is a perfect Bayesian equilibrium.

Problem 7 A buyer and a seller have valuations v_b and v_s. It is common knowledge that there are gains from trade (i.e., that $v_b > v_s$), but the size of the gains is private information, as follows: the seller's valuation is uniformly distributed on $[0, 1]$; the buyer's valuation is $v_b = k \cdot v_s$, where $k > 1$ is common knowledge; the seller knows v_s (and hence v_b) but the buyer does not know v_b (or v_s). Suppose the buyer makes a single offer, p, which the seller either accepts or rejects. What is the perfect Bayesian equilibrium when $k < 2$? When $k > 2$?

Solution:

The extensive-form representation of this game is as follows:



- Clearly, the buyer has no incentive to offer $p > 1$, since the seller will accept $p \geq v_s$ and v_s is uniformly distributed on $[0, 1]$.

- By backward induction, the seller's best response is

$$s_s^*(v_s \mid p) = \begin{cases} \text{accept}, & \text{if } v_s \le p \\ \text{reject}, & \text{if } v_s > p\,. \end{cases}$$

Note that we assume seller will accept if $v_s = p$. This will not affect our analysis of the game since the probability is zero for $v_s = p$.

- The buyer's maximization problem is:

$$\max_{0 \le p \le 1} \mathbb{E}[v_b - p \mid v_s \le p].$$

Since $v_b = kv_s$, the buyer's maximization problem is:

$$\max_{0 \le p \le 1} \int_0^p (kv_s - p)\, dv_s = \max_{0 \le p \le 1} (k/2 - 1)p^2\,.$$

Therefore, the maximizer is

$$p^* = \begin{cases} 1, & \text{if } k > 2 \\ 0, & \text{if } k < 2\,. \end{cases}$$

- Each information set of buyer is reached, so buyer's belief is a uniform distribution on $[0, 1]$.

To summarize, the perfect Bayesian equilibrium is:

$$s_b^* = p^* = \begin{cases} 1, & \text{if } k > 2 \\ 0, & \text{if } k < 2\,, \end{cases}$$

and for $v_s \in [0, 1]$,

$$s_s^*(v_s \mid p) = \begin{cases} \text{accept}, & \text{if } v_s < p \\ \text{accept or reject}, & \text{if } v_s = p \\ \text{reject}, & \text{if } v_s > p\,, \end{cases}$$

the buyer's belief about the seller's valuation is a uniform distribution on $[0, 1]$.